Generalization of Even-Shift Orthogonal Sequences to Multi-Dimension

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Abstract: The even-shift orthogonal sequence (E-sequence) is a binary sequence, whose out-of-phase aperiodic auto-correlation function takes zero at any even shift. This paper considers the generalization of E-sequence to multi-dimension. It is shown that multi-dimensional E-sequences can be constructed by multi-dimensional complementary sequences. Especially a logic function generating multi-dimensional E-sequences of power-of-two length, which can give multi-dimensional E-sequences with a good correlation property that almost 3/4 of all shifts is zero.

1. Introduction

Even-shift orthogonal sequences (E-sequences) are binary sequences whose out-of-phase aperiodic auto-correlation function takes zero at any even shift\(^{[1]}\) - \(^{[4]}\). E-sequences have been applied for spread spectrum communication.

In this paper we consider the generalization of E-sequences to multi-dimension. We clarify the construction of multi-dimensional E-sequences of half length, which mean a pair of binary sequences can be derived from multi-dimensional complementary sequences. Especially a logic function generating multi-dimensional E-sequences with a good correlation property that almost 3/4 of all shifts is zero.

2. Definition of multi-dimensional E-sequences

Let \(e\) be a multi-dimensional \((n\)-dimensional\) binary sequence of length \(L = L_1 \times L_2 \times \cdots \times L_n\), expressed as

\[ e = \{e_{i_1,i_2,\ldots,i_n} \in \{1,-1\} | 0 \leq i_j < L_j\}. \]

The aperiodic auto-correlation function of \(e\) is written as

\[ C_{ee}(\tau_1, \tau_2, \ldots, \tau_n) = \sum_{i_1=0}^{L_1-1} \sum_{i_2=0}^{L_2-1} \cdots \sum_{i_n=0}^{L_n-1} e_{i_1,i_2,\ldots,i_n} e_{i_1+\tau_1,i_2+\tau_2,\ldots,i_n+\tau_n}, \]

where \(e_{i_1,i_2,\ldots,i_n} = 0\) for \(i_j < 0\) and \(i_j \geq L_j\). If the out-of-phase aperiodic auto-correlation function is zero at even shift, i.e.,

\[ C_{ee}(2m_1, 2m_2, \cdots, 2m_n) = \begin{cases} L & \text{for } m_1=m_2=\cdots=m_n=0, \\ 0 & \text{otherwise.} \end{cases} \]

\(e\) is called a multi-dimensional E-sequence, or an \(n\)-dimensional E-sequence.

3. Construction of multi-dimensional E-sequences

In this section, we show that \(n\)-dimensional E-sequences can be constructed from \(n\)-dimensional complementary sequences of half length.

Let \(a\) and \(b\) be \(n\)-dimensional binary sequences of length \(L = L_1 \times L_2 \times \cdots \times L_n\), expressed by

\[ a = \{a_{i_1,i_2,\ldots,i_n} \in \{1,-1\} | 0 \leq i_j < L_j\}, \]

\[ b = \{b_{i_1,i_2,\ldots,i_n} \in \{1,-1\} | 0 \leq i_j < L_j\}. \]

If the sum of their aperiodic auto-correlation functions can be written as

\[ C_{aa}(\tau_1, \tau_2, \ldots, \tau_n) + C_{bb}(\tau_1, \tau_2, \ldots, \tau_n) = \begin{cases} 2L & \text{for } \tau_1 = \tau_2 = \cdots = \tau_n = 0, \\ 0 & \text{otherwise,} \end{cases} \]

the pair of the sequences, \([a,b]\), is called multi-dimensional \((n\)-dimensional\) complementary sequences or complementary arrays \(^{[17]}\). They can be constructed as shown in the following theorems.

[Theorem 1] Let \([a,b]\) be \(n\)-dimensional complementary sequences.

1. Interchanging \(a\) and \(b\) gives an \(n\)-dimensional complementary sequence, i.e., \([b,a]\).
2. Inversion of \(a\) gives an \(n\)-dimensional complementary sequence, i.e., \([-a,b]\).
3. Interchanging some axes gives \(n\)-dimensional complementary sequences, \([a_{i_1,i_2,\ldots,i_n}], \{b_{i_1,i_2,\ldots,i_n}\}\) with \(1 \leq k_m (\neq k_j) \leq n\).
4. Reversing of \(a\) is a mate of \(b\), i.e.,

\([a_{L_1-i_1-1,L_2-i_2-1,\ldots,L_n-i_n-1}], b\].

5. Reversing at some axis gives complementary arrays, \([a_{i_1,\ldots,i_{j-1}-i_j-1,\ldots,i_n}], \{b_{i_1,\ldots,i_{j-1}-i_j-1,\ldots,i_n}\}\).

Note that Theorem 1 can produce a lot of \(n\)-dimensional complementary sequences. For example, use of 2, 3, 2, and 3 following theorems.

[Theorem 2] Let \([a,b]\) be \(n\)-dimensional complementary sequences of length \(L\). We have \((n+1)\)-dimensional complementary sequences \([\tilde{a}, \tilde{b}]\) of length \(2L\), which are written as

\[
\begin{align*}
\tilde{a}_{i_1,i_2,\ldots,i_n,i_n+1} &= \begin{cases} a_{i_1,i_2,\ldots,i_n} & \text{for } i_n+1 = 0, \\ b_{i_1,i_2,\ldots,i_n} & \text{for } i_n+1 = 1, \end{cases} \\
\tilde{b}_{i_1,i_2,\ldots,i_n,i_n+1} &= \begin{cases} a_{i_1,i_2,\ldots,i_n} & \text{for } i_n+1 = 0, \\ b_{i_1,i_2,\ldots,i_n} & \text{for } i_n+1 = 1. \end{cases}
\end{align*}
\]
Multi-dimensional complementary sequences can be easily produced by interleaving and concatenation methods, as well as 1-dimensional complementary sequences [4][5].

[Theorem 3] Let \([a, b]\) be \(n\)-dimensional complementary sequences of length \(L\). We have \(n\)-dimensional complementary sequences \([\hat{a}, \hat{b}]\) of length \(2L\) expressed by

\[
\hat{a}_{i_1, \ldots, i_d} = \begin{cases} a_{i_1, i_2, \ldots, i_d} & \text{for } i'_j = 2i_j, \\ b_{i_1, \ldots, i_d} & \text{for } i'_j = 2i_j + 1, \end{cases}
\]

\[
\hat{b}_{i_1, \ldots, i_d} = \begin{cases} a_{i_1, \ldots, i_d} & \text{for } i'_j = 2i_j, \\ -b_{i_1, \ldots, i_d} & \text{for } i'_j = 2i_j + 1. \end{cases}
\]

[Theorem 4] Let \([a, b]\) be \(n\)-dimensional complementary sequences of length \(L\). We have \(n\)-dimensional complementary sequences \([\hat{a}, \hat{b}]\) of length \(2L\) expressed by

\[
\hat{a}_{i_1, \ldots, i_d} = \begin{cases} a_{i_1, i_2, \ldots, i_d} & \text{for } 0 \leq i'_j < L_j, \\ b_{i_1, \ldots, i_d} - L_j & \text{for } L_j \leq i'_j < 2L_j, \end{cases}
\]

\[
\hat{b}_{i_1, \ldots, i_d} = \begin{cases} a_{i_1, \ldots, i_d} & \text{for } 0 \leq i'_j < L_j, \\ -b_{i_1, \ldots, i_d} - L_j & \text{for } L_j \leq i'_j < 2L_j. \end{cases}
\]

The construction methods in Theorems 3 and 4 are well-known as interleaving and concatenation methods, respectively. We note that \(n\)-dimensional complementary sequences of length \(L = 2^d\) with \(S = 1, 10\) or 26 can be derived from 1-dimensional complementary sequences of length \(S\), called kernels [1][4].

We give a special construction method of \(n\)-dimensional complementary sequences of length \(2^d\), as the following conjecture.

[Conjecture 1] Let \([a, b]\) be \(n\)-dimensional complementary sequences of length \(L = 2^d\). Let \(K = 2^k \leq L\). We have \(n\)-dimensional complementary sequences \([\hat{a}, \hat{b}]\) of length \(2L\) expressed by

\[
\hat{a}_{i_1, \ldots, i_d} = \begin{cases} a_{i_1, i_2, \ldots, i_d} = m_2K + m_1 & \text{for } i'_j = 2m_2K + m_1, \\ b_{i_1, i_2, \ldots, i_d} = m_2K + m_2 + 1 & \text{for } i'_j = (2m_2 + 1)K + m_1, \end{cases}
\]

\[
\hat{b}_{i_1, \ldots, i_d} = \begin{cases} a_{i_1, i_2, \ldots, i_d} = m_2K + m_1 & \text{for } i'_j = 2m_2K + m_1, \\ -b_{i_1, i_2, \ldots, i_d} = m_2K + m_2 + 1 & \text{for } i'_j = (2m_2 + 1)K + m_1, \end{cases}
\]

with \(0 \leq m_1 < K\), \(0 \leq m_2 < L/K \), \(0 \leq i'_j < 2L_j\), and \(0 \leq l_k < L_k\).

Note that combination of Theorems 1-4 and Conjecture 1 can give a lot of \(n\)-dimensional complementary sequences.

[Theorem 5] Let \([a, b]\) be \(n\)-dimensional complementary sequences of length \(L = L_1 \times \cdots \times L_j / 2 \times \cdots \times L_n\) and \([\hat{a}, \hat{b}]\) be \(n\)-dimensional complementary sequences of length \(2L = L_1 \times \cdots \times L_j \times \cdots \times L_n\). Each of \(\hat{a}\) and \(\hat{b}\) given by applying Theorem 3 are E-sequences. It is proved as follows. The aperiodic auto-correlation functions of the above \(n\)-dimensional complementary sequences \(\hat{a}\) and \(\hat{b}\) can be expressed by

\[
C_{\hat{a}\hat{a}}(\tau_1, \tau_2, \ldots, 2m_j, \ldots, \tau_n) = C_{\hat{b}\hat{b}}(\tau_1, \tau_2, \ldots, 2m_j, \ldots, \tau_n) = \left\{ \begin{array}{ll} 2L & (\tau_1 = \tau_2 = \cdots = m_j = \cdots = \tau_n = 0) \\ 0 & \text{otherwise.} \end{array} \right.
\]

Since \(C_{\hat{a}\hat{a}}()\) and \(C_{\hat{b}\hat{b}}()\) are zero if either of \(\tau_1, \tau_2, \ldots, \tau_n\) is even, we have the following theorem.

We show some example for the construct of E-sequence.

[Example 1] Let \(a\) and \(b\) be 2-dimensional complementary sequences of length \(8 \times 2\) as expressed by

\[
a = \left( \begin{array}{cccccccc} + & + & + & - & + & + & + & + \\ + & + & + & - & + & + & + & + \end{array} \right).
\]

\[
b = \left( \begin{array}{cccccccc} + & + & + & + & - & - & - & + \\ - & - & - & - & + & + & + & + \end{array} \right),
\]

where + and − are 1 and −1 respectively. Theorem 3 gives 2-dimensional E-sequences of length \(8 \times 4\).

\[
\hat{a} = \left( \begin{array}{cccccccc} + & + & + & + & - & - & + & + \\ + & + & + & + & - & - & + & + \\ + & + & + & + & - & - & + & + \\ - & - & - & - & + & + & + & + \end{array} \right).
\]

\[
\hat{b} = \left( \begin{array}{cccccccc} + & + & + & + & - & - & + & + \\ + & + & + & + & - & - & + & + \\ + & + & + & + & - & - & + & + \\ - & - & - & - & + & + & + & + \end{array} \right).
\]

The aperiodic auto-correlation functions of \(\hat{a}\) and \(\hat{b}\) can be written as

\[
C_{\hat{a}\hat{a}}(\tau_1, \tau_2) = \left( \begin{array}{cccccccc} 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -3 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 2 & 1 & 0 & 1 & 2 & 1 \\ 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right),
\]

\[
C_{\hat{b}\hat{b}}(\tau_1, \tau_2) = \left( \begin{array}{cccccccc} 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 3 & 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -2 & -1 & 0 & -1 & -2 & -1 \end{array} \right).
\]

[Example 2] Let \(a\) and \(b\) be 3-dimensional complementary sequences of length \(4 \times 4 \times 2\) as expressed by

\[
\]

\[
\]
Theorem 3 gives 3-dimensional E-sequences \([\hat{a}, \hat{b}]\) of length \(4 \times 4 \times 4\)

\[
\hat{a} = \begin{pmatrix}
+ & + & + & - & - & + & + & - \\
+ & + & - & + & + & - & + & - \\
+ & - & + & - & - & + & - & + \\
+ & + & - & - & + & - & + & - \\
\end{pmatrix}
\]

\[
\hat{b} = \begin{pmatrix}
+ & + & + & - & - & + & + & - \\
+ & + & + & - & - & + & + & - \\
+ & + & - & + & + & - & + & - \\
+ & + & - & - & + & - & + & - \\
\end{pmatrix}
\]

The aperiodic auto-correlation functions of \(\hat{a}\) and \(\hat{b}\) are

\[
C_{\hat{a}\hat{a}}(\tau_1, \tau_2, \tau_3) = \begin{pmatrix}
64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 5 & 6 & 1 \\
0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 \\
0 & 0 & 0 & 0 & -1 & -2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & -3 & 0 \\
0 & 0 & 0 & 0 & -2 & -4 & -2 & 0 \\
0 & 0 & 0 & 0 & -1 & -2 & -1 & 0 \\
\end{pmatrix}
\]

\[
C_{\hat{b}\hat{b}}(\tau_1, \tau_2, \tau_3) = \begin{pmatrix}
64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -5 & -6 & -1 \\
0 & 0 & 0 & 0 & -2 & -4 & -2 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & 3 \\
0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\
\end{pmatrix}
\]

by binary expansion of an integer \(i_j(0 \leq i_j < L_j = 2^{\ell_j})\), i.e.,

\[
i_j = i_{j_1}2^0 + i_{j_2}2^1 + \cdots + i_{j_{\ell_j}}2^{\ell_j-1}.
\]

The \(n\)-dimensional E-sequences \(e\) can be written as

\[
e_{i_1, i_2, \ldots, i_j, \ldots, i_n} = (-1)^{f(i_1, i_2, \ldots, i_n)}
\]

The functions \(f()\) is logic function defined by

\[
f(i_1, i_2, \ldots, i_n) = \lambda_1\lambda_2 \oplus \lambda_1\lambda_3 \oplus \cdots \oplus \lambda_{\ell-1}\lambda_\ell
\]

\[
\oplus c_1 i_1 i_{i_1} \oplus c_1,2 i_1,2 i_1,2 \oplus \cdots \oplus c_1 i_1, i_{i_1}, i_{i_1}, i_{i_1}, i_{i_1}
\]

\[
\oplus c_2 i_2 i_{i_2} \oplus c_2,1 i_2,1 i_2,1 \oplus \cdots \oplus c_2 i_2, i_{i_2}, i_{i_2}, i_{i_2}, i_{i_2}
\]

\[
\vdots
\]

\[
\oplus c_{n,1} i_{n,1} \oplus c_{n,2} i_{n,2} \oplus \cdots \oplus c_{n,\ell_n} i_{n,\ell_n} \oplus d,
\]

where \(\oplus\) denotes addition of module 2, i.e., EXOR, \(c_{1,1}, \cdots, c_{n,\ell_n} \in \{0, 1\}\) and \(d \in \{0, 1\}\) are parameter to give different E-sequences, \(\lambda_k(1 \leq k \leq \ell = \sum_{j=1}^{\ell_j} i_j)\) \((j = 1, 2, \ldots, \ell_j)\) of \(i_j\) with \(\lambda_k \neq \lambda_m\) for \(k \neq m\), and either of \(\lambda_1\) or \(\lambda_\ell\) must be \(i_j\).

**Example 3** Let

\[
f(i_1, i_2) = i_{12} i_{11} \oplus i_{11} i_{13} \oplus i_{13} i_{22} \oplus i_{22} i_{21},
\]

be a logic function, where we set that \(\lambda_1 = i_{12} \lambda_2 = i_{12} \lambda_3 = i_{13} \lambda_4 = i_{22} \lambda_5 = i_{21}, c_{1,1} = c_{1,2} = c_{1,3} = c_{2,1} = c_{2,2} = 0\) and \(d = 0\).

**Table 1. The truth table of function (1)**

<table>
<thead>
<tr>
<th>(i_1)</th>
<th>(i_2)</th>
<th>(i_12)</th>
<th>(i_{11})</th>
<th>(i_{10})</th>
<th>(i_{21})</th>
<th>(i_{20})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
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<tr>
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<td>1</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Converting 0 and 1 in Table 1 into + and − respectively gives the 2-dimensional E-sequence in Example 1.

**Example 4** Let a logic function be

\[
f(i_1, i_2) = i_{11} i_{12} \oplus i_{12} i_{13} \oplus i_{13} i_{22} \oplus i_{22} i_{21},
\]

where \(\lambda_1 = i_{j_1}, \lambda_\ell = i_{k_1}(j \neq k)\). We can make following E-sequences of length 8 \(\times 4\).

```
e = \begin{pmatrix}
+ & + & + & + & + & + & + & + \\
+ & + & + & + & + & + & + & + \\
+ & + & + & + & + & + & + & + \\
- & - & - & - & - & - & - & -
\end{pmatrix}
```
which its aperiodic auto-correlation function is written as

\[ C_{ee}(\tau_1, \tau_2) = \begin{pmatrix} 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 5 & 0 & 1 & 0 & 1 \end{pmatrix}. \]

Note that the out-of-phase aperiodic auto-correlation function takes zero if either \( \tau_1 \) or \( \tau_2 \) is even. This is a good correlation property that almost 3/4 of the correlation values is zero. We show one more example of 3-dimensional \( E \)-sequences.

**[Example 5]** If

\[ f(\vec{i}_1, \vec{i}_2, \vec{i}_3) = i_{21}i_{22} \oplus i_{12}i_{22} \oplus i_{22}i_{32} \oplus i_{32}i_{11} \oplus i_{11}i_{31}, \]

E-sequence of length 8 \( \times 4 \) is given as


with

\[ C_{ee}(\tau_1, \tau_2, \tau_3) = \begin{pmatrix} 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & -3 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & -3 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & -3 & 2 & -1 \end{pmatrix}. \]

Note that the out-of-phase aperiodic auto-correlation function takes zero if either \( \tau_2 \) or \( \tau_3 \) is even. So we can produce some \( n \)-dimensional \( E \)-sequences with a good correlation property that almost 3/4 of all shifts is zero.

5. Conclusion

In this paper, we have considered the generalization of \( E \)-sequences to the multi-dimension. We have clarified that \( n \)-dimensional \( E \)-sequences can be constructed from \( n \)-dimensional complementary sequences of half length. We have derived the logic function of \( n \)-dimensional \( E \)-sequences of length \( 2^d \) from the logic function of \( n \)-dimensional complementary sequences. In addition, the logic function can produce multi-dimensional \( E \)-sequences with the good correlation property that almost 3/4 of all shifts is zero.

References


