Time-Domain Asymptotic Solution for Transmitted Gaussian Pulse through a Plane Dielectric Interface

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Abstract—In this paper, we shall derive a time-domain asymptotic solution for the transmitted Gaussian pulse wave observed in the rarer medium when the Gaussian pulse is incident on a plane dielectric interface from the denser medium. The validity of the time-domain asymptotic solution is confirmed by comparing with the reference solution calculated numerically. We will show that the evanescent pulse wave and the lateral wave type transmitted pulse wave are observed in the far and shallow region.

I. INTRODUCTION

The reflection from, the scattering by, and the transmission through a plane dielectric interface of the cylindrical, spherical, and Gaussian beam waves have been the important research subjects [1]-[8] in the area of the electromagnetic theory, the antennas and propagation, the nondestructive measurement, the metamaterial, and so on.

Recently, we have derived the frequency-domain uniform asymptotic solution for the transmitted wave through the plane dielectric interface from the denser medium [7], [8]. The uniform asymptotic solution can connect the asymptotic solution in the near region and that in the far region through the transition region [7], [8]. We have also derived the solution for the lateral wave type transmitted wave observed in the rarer medium which is very similar to the well-known lateral wave observed in the denser medium [1]-[8].

In this paper, we shall derive the time-domain asymptotic solution for the transmitted Gaussian pulse wave observed in the rarer medium when the Gaussian pulse wave is incident on the plane dielectric interface from the denser medium. The validity of the time-domain asymptotic solution is confirmed by comparing with the reference solution obtained from the numerical evaluation of the integral representation for the transmitted Gaussian pulse wave. We will show that the evanescent pulse wave and the lateral wave type transmitted pulse wave are observed in the denser medium.

II. TIME-DOMAIN ASYMPTOTIC SOLUTION FOR TRANSMITTED GAUSSIAN PULSE WAVE

A. Frequency-Domain Asymptotic Solution

1) Formulation and Integral Representation: Fig. 1 shows the transmission and scattering of the cylindrical wave through a plane dielectric interface and the Cartesian coordinate system \((x, y, z)\). The observation area in the rarer medium with the permittivity \(\varepsilon_2\) and the permeability \(\mu_0\) may be divided into the near region, the transition region [8], and the far region in the \(x\)-direction, and into the shallow region and the deep region in the \(z\)-direction. We assume that the medium 1 \((\varepsilon_1, \mu_0)\) is denser than the medium 2 \((\varepsilon_2, \mu_0)\), i.e., \(\varepsilon_1 > \varepsilon_2\).

When the electromagnetic wave is radiated from the electric line source \(Q(x, z) = Q(0, -h)\) directing the \(y\)-direction, the electric field \(E_y(x, z)\) observed in the medium 2 \((\varepsilon_2, \mu_0)\) may be given by [8], [9]

\[
E_y = i\omega\mu_0 G, \quad G = \frac{i}{4\pi} \int_{P_0} T(\theta)e^{ikq(\theta)}d\theta
\]  

(1)

\[
T(\theta) = \frac{2\cos \theta}{\cos \theta + \sqrt{n^2 - \sin^2 \theta}}
\]  

(2)

\[
q(\theta) = R\cos(\theta - \theta_0) + z\sqrt{n^2 - \sin^2 \theta}
\]  

(3)

In the above equation, \(T(\theta)\) corresponds to the transmission coefficient, \(\omega\), \(I\), and \(n(= \sqrt{\varepsilon_2/\varepsilon_1})\) denote the angular frequency, the line current, and the refractive index, respectively. Notations \(R\) and \(\theta_0\) are defined geometrically in Fig. 1. The original integration path \(P_0\) in (1) is shown in the complex \(\theta\)-plane in Fig. 2.
2) Asymptotic Solution in Near and Transition Regions:
When the observation point \( P_1 \) is located in the near region and the transition region between the dotted curves \( AG \) and \( AH \) in Fig. 1, one may derive the uniform asymptotic solution for the transmitted and scattered wave from (1) as follows [7], [8]

\[
G = G_{go} + G_{tran}
\]

where \( G_{go} \) is the contribution from the steepest descent path \( SDP_{th} \) in Fig. 2 given by [7]-[9].

\[
G_{go} = \frac{i}{4\pi} T(\theta_s) \sqrt{\frac{2\pi}{k_1 q''(\theta_s)}} e^{ik_1 L_1 + ik_2 L_2 - i\pi/4}
\]

(5)

The solution shows the transmitted geometrical ray \( Q \rightarrow C \rightarrow P_1 \) in Fig. 1. The saddle point or the incident angle \( \theta_s \) (see Fig. 1) is determined from the saddle point equation \((\partial T/\partial \theta)q(\theta) = 0\). The geometrical parameters \( l_1 \) and \( l_2 \) are defined in Fig. 1.

While \( G_{tran} \) in (4) denotes the transition wave [8]-[10] expressed as

\[
G_{tran} = \hat{G} - G_{go}, \quad \hat{G} = \frac{i}{4\pi} \int_{P_{Go}} T(\theta) e^{ik_1 q(\theta)} d\theta
\]

(7)

In \( \hat{G}, P_{Go} \) is the new integration path shown in Fig. 2. The asymptotic solution for \( \hat{G} \) in (7) has been studied extensively in [8], [9].

When the observation point moves from the transition region to the near region, \( \hat{G} \) approaches the geometrical ray solution \( G_{go} \). Thus, it is clear that the transition wave plays an important role in the transition region.

3) Asymptotic Solution in Transition and Far Regions:
When the observation point \( P_2 \) is located in the transition region between the dotted curves \( AH \) and \( AH' \) in the far region in the shallow region (see Fig. 1), the two saddle points \( \theta_s \) and \( \theta_\nu \) contribute to the integral. Thus, the transmitted and scattered wave \( G \) in (1) may be represented as

\[
G = \hat{G} + G_{eva}
\]

(8)

where \( \hat{G} \) denotes the integral along the steepest descent path \( SDP_{th} \) and \( G_{eva} \) denotes the integration along the steepest descent path \( SDP_{th} \) (see Fig. 3).

The asymptotic solution for \( \hat{G} \) consisting of the lateral wave transmitted ray \( G_{lat.wave} \) and the transition wave \( G_{tran} \) may be given by [7]-[9]

\[
\hat{G} \sim G_{tran} + G_{lat.wave}
\]

(9)

\[
G_{lat.wave} = \frac{i}{4\pi k_1} \frac{2\pi}{\sqrt{n^2 - \sin^2 \theta_\nu}} e^{ik_1 L_1 + ik_2 L_2}
\]

(10)

where the propagation distances \( L_1 \) and \( L_2 \) of \( G_{lat.wave} \) behaves like a lateral wave which is observed in the denser medium 1 [1]-[5]. The asymptotic solution for the transition wave \( G_{tran} \) has been derived in [8], [9]. The transition wave \( G_{tran} \) plays an important role in the transition region between the curves \( AH \) and \( AH' \) and approaches zero \( (G_{tran} \to 0) \) as the observation point moves to the far region.

While the evanescent wave \( G_{eva} \) in (8) may be represented by [8], [9]

\[
G_{eva} \sim \frac{i}{4\pi k_1} \frac{2\pi}{\sqrt{n^2 - \sin^2 \theta_\nu}} e^{ik_1 R - i\pi/4} \sqrt{T(\theta_0)} e^{ik_2 z - k_2 z} \sin^{-\nu} \delta
\]

(11)

where \( R(= QE) \) and \( \theta_0 \) are defined geometrically in Fig. 1. The evanescent wave \( G_{eva} \) decay exponentially in the \( z \)-direction as shown in (11) and Fig. 1.

Substituting (9) associated with (10) and (11) into (8), one may obtain the uniform asymptotic solution applicable in the transition region between \( AH \) and \( AH' \) and in the far region.

B. Time-Domain Asymptotic Solution

The time-domain transmitted and scattered wave solution \( y(x, z, t) \equiv y(t) \) can be obtained by utilizing the inverse Fourier transform of the product of the frequency-domain solution \( E(x, y, \omega) \equiv G(\omega) \) and the frequency spectrum \( S(\omega) \) of the source function \( s(t) \) as follows [10], [11]

\[
y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) G(\omega) e^{-i\omega t} d\omega
\]

(12)
In obtaining (12), the normalization $i\omega_0 I = 1$ has been applied. Here, we will assume the Gaussian-type modulated pulse source $s(t)$ given by (10), (11).

$$s(t) = \begin{cases} e^{-i\omega_0 (t - t_0)} e^{-\left(\frac{t - t_0}{\tau_0}\right)^2}, & 0 < t \leq 2t_0 \\ 0, & t < 0, \ t > 2t_0 \end{cases}$$

(13)

The frequency spectrum of $s(t)$ is given by (10), (11)

$$S(\omega) = 2d\sqrt{\pi}e^{i\omega_0 t_0 - d^2(\omega - \omega_0)^2} \text{Re}[\text{erf}\{\beta(\omega)\}]$$

(14)

$$\beta(\omega) = \frac{t_0}{2d^2} - i\omega(\omega - \omega_0)$$

(15)

where erf denotes the error function (10), (11). Figs. 4(a) and 4(b) show the real part of the source function $s(t)$ and the magnitude of the Fourier spectrum $S(\omega)$, respectively.

As shown in Fig. 4(b), the main portion of the spectrum $|S(\omega)|$ is distributed around the central frequency $\omega_0 = 2\pi \times 10^9$ from $\omega = \omega_0$ to $\omega = \omega_2$ in the high frequency domain. Therefore, the high frequency asymptotic solutions obtained in the Section II-A can be used in (12).

The asymptotic solutions in the Section II-A may be represented by

$$G_j(\omega) = A_j(\omega) e^{i\tau_j \omega}$$

(16)

where the subscript "$j$" may be "go" (geometrical ray) and "tran" (transition wave) in (4) and "lat.wave" (lateral wave type transmitted wave), "tran" (transition wave), and "evan" (evanescent wave) in (8). For example, when $j \equiv go$, (16) implies that $G_{go}(\omega) \equiv A_{go}(\omega) e^{i\omega\tau_{go}}$ where $G_{go}$ denotes the transmitted geometrical ray solution derived in (5) associated with (6).

The complex amplitude $A_j(\omega)$ is the slowly varying function of $\omega$ and the propagation time $\tau_j$ are defined by

$$\tau_{go} = \frac{l_1}{c_1} + \frac{l_2}{c_2}, \ \tau_{lat.wave} = \frac{L_1}{c_1} + \frac{L_2}{c_2}, \ \tau_{evan} = \frac{R}{c_1}$$

(17)

where $c_1$ and $c_2$ are the propagation speeds in the medium 1 and the medium 2, respectively.

Substituting $G_j(\omega)$ in (16) for $G(\omega)$ in (12) yields

$$y_j(t) = \frac{d}{\sqrt{\pi}} e^{-i\omega_0 T_j - \tau_j^2/(4d^2)} I_j(\Omega)$$

(18)

$$I_j(\Omega) = \int_{-\infty}^{\infty} F_j \left( \frac{\omega'}{2d^2} \right) e^{-\Omega h_j(\omega')} \frac{d\omega'}{2d^2}, \ \Omega = \frac{1}{4d^2}$$

(19)

$$T_j = t - t_0 - \tau_j, \ F_j(\omega) = A_j(\omega) \text{Re}[\text{erf}\{\beta(\omega)\}]$$

(20)

$$h_j(\omega') = (\omega' - \omega_0^2 + iT_j)^2, \ \omega_0 = 2d^2\omega_0$$

(21)

In obtaining above representations, the transformation from $\omega$ to $\omega'$ via $\omega' = 2d^2\omega_0$ has been applied.

The integral $I_j(\Omega)$ in (19) can be evaluated asymptotically by applying the saddle point technique (12). Then the asymptotic solution for $I_j(\Omega)$ is substituted into (18) to obtain the time-domain asymptotic solution as follows.

$$y_j(t) = A_j(\omega_j) \text{Re}[\text{erf}\{\beta(\omega_j)\}] s(t - \tau_j)$$

(22)

$$\omega_j = \omega_0 - i\frac{T_j}{2d^2}$$

(23)

From (22), it is clarified that the pulse wave $s(t)$ radiated from the source $Q(0, -h)$ is observed after the propagation time $\tau_j$ of the transmitted and scattered pulse wave.

In the Section III, we will examine the validity and the physical interpretation of the time-domain asymptotic solution in (22) associated with (23).

III. NUMERICAL RESULTS AND DISCUSSIONS

We have performed extensive numerical calculations to confirm the validity of the time-domain asymptotic solutions proposed in this study. Here we will show only a few typical results since other results are similar to those given here.

Fig. 5 shows the transmitted pulse wave observed at the point $P_1$ located in the near region (see Fig. 1) when the Gaussian-type modulated pulse wave shown in Figs. 4(a) and 4(b) is radiated from the electric line source $Q(0, -h)$. The transmitted pulse wave is calculated by using the time-domain geometrical ray solution $y_{go}(t)$ obtained from (22) with the replacement of the subscript "go" by "go" and is shown by the blue solid curve ( ). To assess the validity of the time-domain geometrical ray solution, we have obtained...
In Figs. 6(a) and 6(b), we have shown the time-domain transmitted wave received at the observation point $P_2$ located in the far and shallow region (see Fig. 1). In this region, one may receive the lateral wave type transmitted pulse wave and the evanescent pulse wave as shown in (8) associated with (9) and (10). Note that the transition wave in (9) is negligibly small in the far region. Fig. 6(a) corresponds to the lateral wave type transmitted pulse wave while Fig. 6(b) corresponds to the evanescent pulse wave. It is shown in Fig. 6(a) that the calculation result obtained from the lateral wave type transmitted pulse wave $y_{lat.wave}(t)$ (---) in (22) with "j" replaced by "lat.wave" agrees excellently with the reference solution.

While, Fig. 6(b) shows that the time-domain asymptotic solution $y_{eva}(t)$ for the evanescent pulse wave (----) (see (22)) agrees excellently with the reference solution calculated from the numerical integration of the integrals (see (1) and (12)).

It is interesting to note that, due to the difference of the optical path lengths of the lateral wave type transmitted pulse wave $y_{lat.wave}(t)$ and the evanescent pulse wave $y_{eva}(t)$, these two pulse waves are observed separately.

**IV. Conclusion**

In this paper, we have derived the time-domain asymptotic solutions for the transmitted pulse waves observed in the rarer medium when the Gaussian-type modulated pulse wave is incident on a plane dielectric interface from the denser medium. We have confirmed the validity of the time-domain asymptotic solutions by comparing with the reference solution calculated numerically. We have shown the interesting phenomenon that the lateral wave type transmitted pulse wave arrives at the observation point earlier than the evanescent pulse wave. It has been shown that these two pulses are observed separately.

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**References**