Optimal currents on small antennas using convex optimization

Mats Gustafsson

Department of Electrical and Information Technology, Lund University
Box 118, SE-221 00 Lund, Sweden
mats.gustafsson@eit.lth.se

Abstract—In this paper, it is shown that convex optimization can be used to formulate and solve several fundamental problems in antenna theory. The strategy is based on reformulations of the antenna problem as convex optimization problems, where it is used that the radiated power, stored energy, dissipated power, and radiated field are convex functions in the current density. This leads to a systematic and computational effective formulation for analysis of optimal currents and bounds on e.g., $G/Q$, and $Q$ for superdirective antennas, and $Q$ for prescribed radiation patterns. It also enables analysis of antennas that are embedded in devices.

1. Introduction

Physical bounds on small antennas were first studied by Wheeler [1] and Chu [2] more than 60 years ago. The approach by Chu [2] has dominated the research until recently, see [3] for an overview. A new approach was presented in [4], [5] that is valid for arbitrary shapes and hence circumvents the previous restriction to spheres [2], see also [6]–[8]. In [9], it is further shown that optimal current and charge distributions on the antenna structure can be analyzed.

Here, we follow the approach in [10], where various fundamental antenna problems are analyzed using convex optimization. Convex optimization is advantageous as there are; a well developed theory, several efficient solvers, and explicit error estimates [11]. In many cases one can even consider a problem as solved if it is formulated as a convex optimization problem. We use the results by Vandenbosch [12] to express the stored electric and magnetic energies as quadratic forms in the current density. The energy forms are positive semidefinite for small antennas [9] and hence convex, see also [13] for an alternative interpretation of the stored energies. Here, the results are restricted to passive and linear material models.

2. Stored energy and radiation intensity

We follow the approach in [9], [10] and use the results by Vandenbosch [12], to express the stored electric energy as a quadratic form in the current density $J$, i.e., $W_e = \frac{\mu_0}{16\pi^2} w_e^{(e)}$, where

$$w_e^{(e)}(J) = \int_V \int_V \nabla_1 \cdot J_1 \nabla_2 \cdot J_2 \frac{\cos(kR_{12})}{R_{12}} - \frac{k}{2} (k^2 J_1 \cdot J_2^* - \nabla_1 \cdot J_1 \nabla_2 \cdot J_2^*) \sin(kR_{12}) \, dV_1 \, dV_2,$$

and $J_1 = J(r_1)$, $J_2 = J(r_2)$, $R_{12} = |r_1 - r_2|$, $k$ the wavenumber, and $\mu_0$ is the permeability of free space. The stored electric energy (1) can be interpreted as the difference between the electric energy density and the energy density of the far field for many currents [13], i.e.,

$$W_e = \frac{\epsilon_0}{4} \int_{\mathbb{R}^3} |E(\hat{r})|^2 - \frac{|E_\infty(\hat{r})|^2}{r^2} \, dV,$$

(2)

where $E_\infty = \lim_{r \to \infty} r^0 e^{jkr} E$ denotes the electric far field, $r = |\hat{r}|$, $\hat{r} = r/r$, $\epsilon_0$ the permittivity of free space, and the integration is over an infinite spherical volume. The stored magnetic energy is $W_m = \frac{\mu_0}{16\pi^2} w_m^{(m)}$, where

$$w_m^{(m)} = \int_V \int_V \frac{k}{2} (k^2 J_1 \cdot J_2^* - \nabla_1 \cdot J_1 \nabla_2 \cdot J_2^*) \sin(kR_{12}) \, dV_1 \, dV_2.$$  (3)

The corresponding expression for the total radiated power is

$$P_{\text{rad}} = \frac{n_0}{8\pi} P_{\text{rad}} \quad \text{with}$$

$$P_{\text{rad}} = \int_V \int_V \left( k^2 J_1 \cdot J_2^* - \nabla_1 \cdot J_1 \nabla_2 \cdot J_2^* \right) \frac{\sin(|k| r_1 - r_2)}{|r_1 - r_2|} \, dV_1 \, dV_2$$

(4)

and $n_0$ the impedance of free space. We also use the partial radiation intensity in a direction $\hat{k}$ and for the polarization $\hat{e}$, i.e., $P(\hat{e}, \hat{k}) = \frac{n_0 k^2}{32\pi^2} \hat{e}^* \cdot \hat{F}(\hat{k})$, where

$$\hat{e}^* \cdot \hat{F}(\hat{k}) = \int_V \hat{e}^* \cdot J(r) e^{jk \cdot r} \, dV.$$ (5)

The normalized quantities, $w_e^{(e)}$, $w_m^{(m)}$, in (3) and $p_{\text{rad}}$ in (4) are introduced to simplify the optimization approach used in this paper.

3. MoM formulation and convex optimization

We expand the current density in local basis functions $J(r) = \sum_n J_n \psi_n(r)$ similar to the method of moments (MoM). This transforms the stored electric energy to the quadratic form

$$w_e^{(e)} \approx \sum_{mn} J_n^* X_{e,mn} J_m = J^H X_e J,$$

where $X_e$ is the matrix associated with (1) and $J$ a column matrix with elements $J_n$, see [10]. We define similar matrices for the stored magnetic energy (3), $X_m$, radiated power (4), $R_e$, and radiated far field (5) $F$. 

90
We can now formulate several convex optimization problems for the antenna performance. The case with maximal gain Q-factor quotient is [10]

$$\begin{align*}
\text{minimize} & \quad \max \{J^H X_c J, J^H X_m J\} \\
\text{subject to} & \quad \text{Re}[F^H J] = 1 \\
\end{align*}$$

where we used that $G/Q$ is invariant for multiplicative scaling $J \rightarrow \alpha J$ [9], [10]. The solution to this convex optimization is similar to the results from the forward scattering approach in [4], [5]. It is easy to implement the convex optimization problem (6) using e.g., matlab and CVX [14] as

```matlab
cvx_begin
variable J(n) complex;
variables W

We: quad_form(J,Xe) <= W;
Wm: quad_form(J,Xm) <= W;
Fp: real(F'*J) <= -1;

minimize W
subject to Re[F^H J] = 1,
J^H R_c J \leq k^3 D_0^{-1},
```

Convex optimization is a convenient tool to formulate the antenna problems as it is flexible and can be used to analyze many different antenna problems. We can for example analyze the minimum Q of superdirective antennas by adding a constraint on the total radiated power to the optimization problem (6). With the constraint $P_{\text{rad}} \leq 4\pi P(k,e) D_0^{-1}$, we get

$$\begin{align*}
\text{minimize} & \quad \max \{J^H X_c J, J^H X_m J\} \\
\text{subject to} & \quad \text{Re}[F^H J] = 1, \\
& \quad J^H R_c J \leq k^3 D_0^{-1},
\end{align*}$$

where the factor $k^3 D_0^{-1}$ is due to the normalization of $R_c$ and $F$. We also note that the equality constraint $\text{Re}[F^H J] = 1$ can be replaced with the inequality $\text{Re}[F^H J] \leq -1$ giving the implementation

```matlab
cvx_begin
variable J(n) complex;
variables W

We: quad_form(J,Xe) <= W;
Wm: quad_form(J,Xm) <= W;
Fp: real(F'*J) <= -1;
Pr: quad_form(J,Rr) <= k^3/D0;

minimize W
subject to Re[F^H J] = 1,
J^H R_c J \leq k^3 D_0^{-1},
```

4. Numerical Example

A planar rectangle with side lengths $\ell_x$ and $\ell_y/2$ is used to illustrate the bound for superdirective endfire antennas. The $Q$ is determined for the directivities $D(\hat{x}) \geq D_0 = \{3, 5, 7, 9\}$. It is seen that $Q$ increases with $D_0$.

4. Conclusions

It is shown that convex optimization offers a framework to analyze several fundamental problems for small antennas. The approach is based on expressing the stored energy and radiated power as quadratic forms in the current density [12], [13].

**REFERENCES**


