Application of Lattice to Convolutional Codes: Signal Codes and Turbo Signal Codes
（格子の畳込み符号への応用）

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Outline

• A very brief review of lattices
• Convolutional lattice codes (Signal codes)
• Recursive convolutional lattice codes and their parallel concatenation (Turbo signal codes)
• Some performance comparison
• Conclusions
Lattice (1)

- Let $\mathbf{g}_1$ and $\mathbf{g}_2$ denote the $n$-dimensional (real-valued) vectors that are linearly independent.
  - $\mathbf{g}_1$ and $\mathbf{g}_2$ are called the basis vectors of the $n$ (real) dimensional Euclidean space.

**Example (n = 2):**

$$n = 2$$

$$\mathbf{g}_1 = (1 \ 0)^T, \quad \mathbf{g}_2 = (0 \ 1)^T$$

- $\mathbf{g}_1$ and $\mathbf{g}_2$ span the two-dimensional Euclidean space.

**Example (cos 30° sin 30°):**

$$\mathbf{g}_1 = \left( \cos 30^\circ \sin 30^\circ \right)^T,$$

$$\mathbf{g}_2 = \left( \cos 90^\circ \sin 90^\circ \right)^T$$

- $\mathbf{g}_1$ and $\mathbf{g}_2$ span the two-dimensional Euclidean space with a diagonal orientation.
Lattice (2)

- Two-dimensional lattice $\Lambda$ is given by a set of real-valued points that are specified by linear combination of the basis vectors with integer coefficient.

$$\Lambda = \left\{ \lambda = \sum_{k=1}^{2} a_k g_k : a_k \in \mathbb{Z} \right\}$$

$$= \left\{ \lambda = G a : a \in \mathbb{Z}^2 \right\} \text{ where } G = \begin{pmatrix} g_1 & g_2 \end{pmatrix}$$

$\Rightarrow$ G is a generator matrix of lattice

Note that $\text{rank}(G) = 2$, $\det(G) \neq 0$
Lattices and Sublattices

- For a given lattice $\Lambda$, its subset $\Lambda' \subset \Lambda$ is called sublattice.

\[
\Lambda = \mathbb{Z}^2, \quad \Lambda' = 2\mathbb{Z}^2, \quad \Lambda'' = 4\mathbb{Z}^2
\]
Partition of Lattice

- $\Lambda'$ induces a partition of $\Lambda$, i.e., $\Lambda / \Lambda'$, into equivalent classes modulo $\Lambda'$ (quotient group).
- Each equivalent class is called a coset of $\Lambda'$.
- The number of the cosets, $|\Lambda / \Lambda'|$, is called the order of the quotient group.

**cosets of $\Lambda'$ for $\mathbb{Z}^2/2\mathbb{Z}^2$**

- $\Lambda' + (0,0)$
- $\Lambda' + (1,0)$
- $\Lambda' + (0,1)$
- $\Lambda' + (1,1)$

**coset representative**
A Practical Example of Lattice Codes

- Ungerboeck’s 2-D 64-QAM TCM (coset code):

1 bit \rightarrow \text{rate-1/2 binary convolutional encoder} \rightarrow \text{2 bits} \rightarrow \text{select a coset}

4 bits \rightarrow \text{select a lattice point}

\[ |\mathbb{Z}^2/2\mathbb{Z}^2| = 4 \]
Coset Decomposition

- If we denote a set of coset representatives by $[\Lambda/\Lambda']$, then each lattice point $\lambda$ of $\Lambda$ is expressed with respect to the point $\lambda'$ of the sublattice $\Lambda'$ as

$$\lambda \in \Lambda, \lambda' \in \Lambda' \Rightarrow \lambda = \lambda' + c, \quad c \in [\Lambda/\Lambda']$$

or alternatively

$$\Lambda = \Lambda' + [\Lambda/\Lambda']$$

$$\mathbb{Z}^2 = 2\mathbb{Z}^2 + \{(0,0),(0,1),(1,0),(1,1)\}$$
Signal Codes (1)

- Convolutional lattice codes (Shalvi 2011)
- Input: $L^2$-QAM constellation (uncoded)
- For memory size $P$, constraint length $K = P + 1$

\[ P = 2 \quad L = 4 \]

\[ L^2P \quad \text{States} \]
Signal Codes (2)

- Lattice structure

Example: $P = 2, \ k = 3, \ n = k + P = 5$

$$\Lambda = \left\{ \lambda = G \mathbf{a} : \mathbf{a} \in (\mathbb{Z}_L [j])^3 \right\}$$

where

$$G = \begin{pmatrix}
1 & 0 & 0 \\
g_1 & 1 & 0 \\
g_2 & g_1 & 1 \\
0 & g_2 & g_1 \\
0 & 0 & g_2 \\
\end{pmatrix}$$

$g_1, g_2 \in \mathbb{C}$
Signal Codes (3)

Average Power:

\[ P_{\text{av}} = E \left\{ |\tilde{x}_i|^2 \right\} = 1 + \sum_{k=1}^{2} |g_k|^2 \]

FIR tap

\[ G(z) = \left(1 - z_0 z^{-1} \right)^P \]

\[ z_0 = -0.90 e^{j0.12\pi}, \quad P = 2 \]

※ Poles close to a unit circle yield high coding gain

\[ \Rightarrow \quad P_{\text{av}} = 6.9 \text{dB} \]

Average power reduction by shaping is essential.
Signal Codes (4)

The number of trellis states is unbounded!
Recursive Convolutional Lattice Code (RCLC)

- Convolutional lattice codes with constrained states (Mitran 2015)
- Recursive form of RCLC: describes the state boundedness

State is defined by the output symbol.

$L^2$ points

$L^2N_{bv}$ points

Trellis State

$P = 2$

$L^2N_{bv}P$ states

$\tilde{x}_i$

$L = 2$

$\mathbb{Z}[j]$
Constellation Design (1)

We consider the following formal power series:

\[
\mathbb{Z}[\omega] := \left\{ a_0 + a_1 \omega + a_2 \omega^2 + \cdots + a_{n-1} \omega^{n-1} : a_k \in \mathbb{Z} \right\}, \quad \omega = e^{j\pi/n}
\]

which has the following ring property (note: \( \omega^n = -\omega \))

\[
\alpha, \beta \in \mathbb{Z}[\omega] \implies \alpha + \beta \in \mathbb{Z}[\omega], \quad \alpha \beta \in \mathbb{Z}[\omega]
\]

Since \( e^{j\pi/2} = j \), for even values of \( n = 2N_{bv} \), we have

\[
\mathbb{Z}[\omega] := \left\{ a_0 + b_0 j + (a_1 + b_1 j) \omega + \cdots + (a_{N_{bv}-1} + b_{N_{bv}-1} j) \omega^{N_{bv}-1} : a_k, b_k \in \mathbb{Z} \right\}
\]

\[
= \left\{ c_0 + c_1 \omega + \cdots + c_{N_{bv}-1} \omega^{N_{bv}-1} : c_k \in \mathbb{Z}[j] \right\}, \quad \omega = e^{j\pi/2N_{bv}}
\]
Constellation Design (2)

To limit the size of signal points, we further put a constraint that $a_k$ and $b_k$ are integer rings (i.e., coset leaders of the following quotient):

$$C (L, N_{bv}) := \mathbb{Z} [\omega] / L \mathbb{Z} [\omega]$$

$$= \left\{ a_0 + b_0 j + (a_1 + b_1 j) \omega + \cdots + \left( a_{N_{bv}-1} + b_{N_{bv}-1} j \right) \omega^{N_{bv}-1} : a_k, b_k \in \mathbb{Z}_L \right\}$$

$$= \left\{ c_0 + c_1 \omega + \cdots + c_{N_{bv}-1} \omega^{N_{bv}-1} : c_k \in \mathbb{Z}_L [j] \right\}, \quad \omega = e^{j \pi / 2 N_{bv}}$$

$$C (L, N_{bv}) \subseteq \mathbb{Z} \left[ e^{j \pi / 2 N_{bv}} \right]$$
Constellation Design (3)

In general, for $N_{bv} > 1$, we have

$$C(L, 1) \subset C(L, N_{bv}) \subset C(L + 1, N_{bv}) \subset \mathbb{Z}[e^{j\pi/2N_{bv}}]$$
State-Constrained Signal Codes

\[ x_i = x'_i + L b_i \subset C(L, N_{bv}) \]

\[ x'_i = a_i + h_1 x_{i-1} + h_2 x_{i-2} \subset \mathbb{Z} \left[ e^{j\pi/2N_{bv}} \right] \]

\[ x_{i-1}, x_{i-2} \subset C(L, N_{bv}) \]

\[ C(L, N_{bv}) \]

\[ C(L, 1) \]

Trellis State

DC Shift

\[ x_i \]
Constrained Capacity

Information rate (bits): $\log_2 |C(L, 1)| = \log_2 L^2 = 2 \log_2 L$

Constellation size: $\log_2 |C(L, N_{bv})| = \log_2 L^{2N_{bv}} = 2N_{bv} \log_2 L$
Some Remarks

- The information rate is \(2 \log_2 L\) bits per 2D.
- Code (tap) selection: Unlike the conventional lattice codes, it is difficult to analyze minimum Euclidean distance due to the lack of regularity, resorting to brute-force search.
Turbo Signal Codes

- By constraining the state size \( (L^{2N_{bv}})^P \) states, trellis based decoding such as Viterbi algorithm can be now employed.
- However, the performance may not be significant because it is a variant of recursive convolutional codes.
- Due to the limited number of states, the BCJR decoding can be used.
- Since the code is recursive, can we take an approach similar to binary turbo codes and perform MAP decoding?
Turbo Signal Codes

\[ C(L, N_{bv}) \]

\[ f_0 \times \]

\[ f_1 \times \]

\[ f_2 \times \]

\[ x_i \]

\[ h_1 \times \]

\[ h_2 \times \]

\[ x_{i-1} \]

\[ x_{i-2} \]

\[ \pi \]

\[ a_i \]

\[ L b_i \]

\[ u_i \]

\[ \tilde{u}_i^1 \]

\[ \tilde{u}_i^2 \]

\[ \text{DC Shift} \]

RCLC 1

RCLC 2

\[ C(L, 1) \]
Comparison with Non-binary LDPC

Lattice: $P = 1, L = 2, N_{bv} = 2$

LDPC: QPSK, rate = 1/2

(frame length 4096 bits)
Comparison of Decoding Complexity

松峯, 落合, RCS研究会 (2015年8月)
Conclusions

• We have considered application of lattice in an unconventional scenario:
  – Signal Codes
  – Turbo Signal Codes
• Due to the lack of structure, optimal design of coding (filter taps) is challenging from a theoretical viewpoint
• There are many issues unknown:
  – Performance analysis and code design
  – Extension to even higher constellation size
  – Complexity vs. performance trade-off
  – Puncturing for higher rate