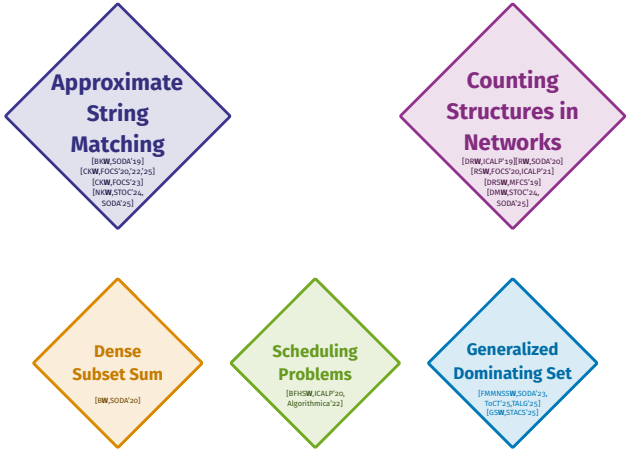
Revitalizing Research on Approximate String Matching Algorithms

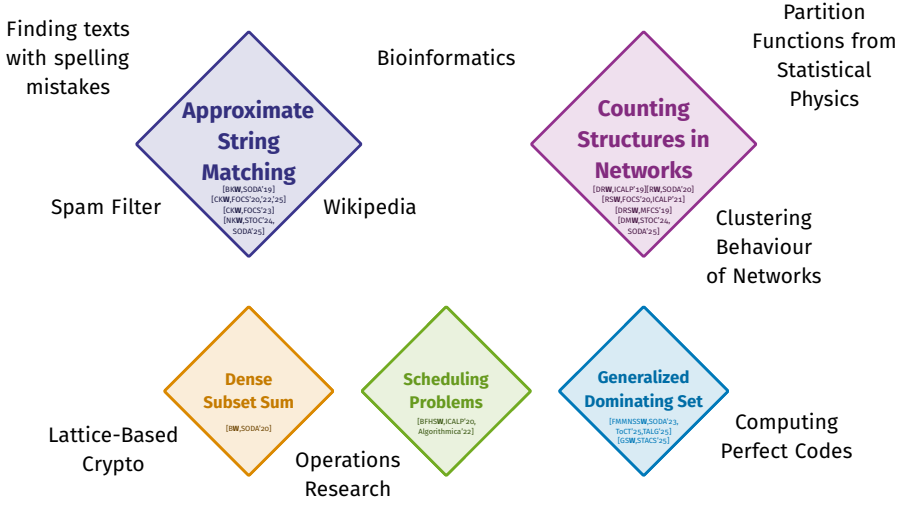
Philip Wellnitz
National Institute of Informatics

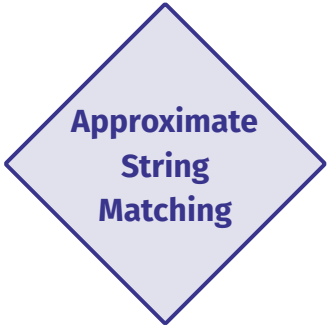
Based on joint works with Karl Bringmann, Alejandro Cassis, Panagiotis Charalampopoulos, Tomasz Kociumaka, Marvin Künnemann, and Jakob Nogler.

Research Focus: Theoretical Guarantees for Fundamental Problems



Research Focus: Theoretical Guarantees for Fundamental Problems of Practical Relevance




Approximate
String
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An Example

...

(TU Kaiserslautern).

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(University of Pennsylvania), (University of Pennsylvania), (University of Pennsylvania).

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early 1980's

Approximate String Matching

Given a text T, a pattern P, and an integer k, identify the (starting positions of) substrings of T that are at **edit distance** of at most k to P.

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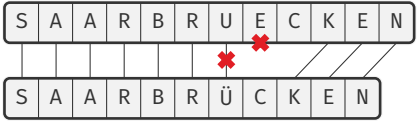
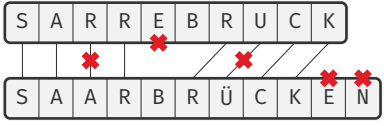
Edit distance: minimum number of insertions, deletions, or substitutions of single characters to transform one string into another string





Outline

Basic Tricks and Tools, Previous Work

New Algorithms via Structural Insights

Spotlight Extension: Quantum Algorithms for ASM

Spotlight Extension: String Matching with Weighted Edits

Open Problems and Future Directions

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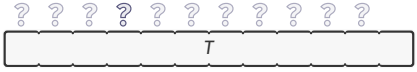

Focus: Obtain starting positions of occurrences

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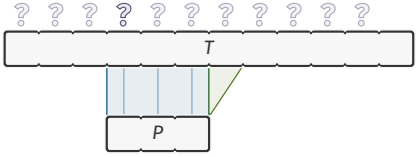

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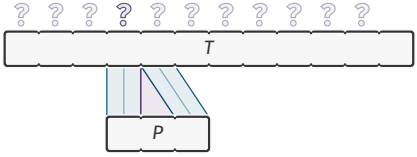

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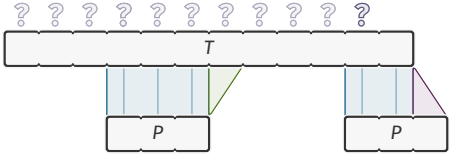

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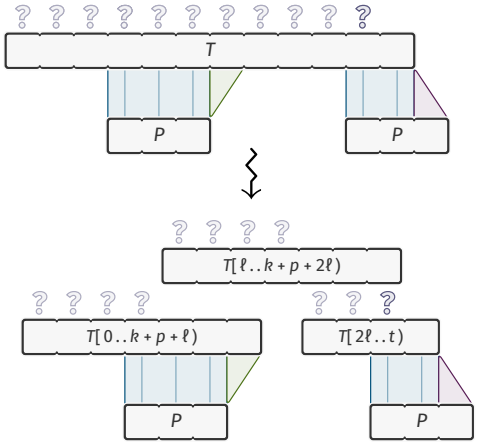
"Standard Trick": write $t := |T|$, $p := |P|$
Split T into overlapping fragments of len $\ell + p + k$

$\rightsquigarrow O(t/\ell)$ instances,
each "responsible" for its first ℓ positions



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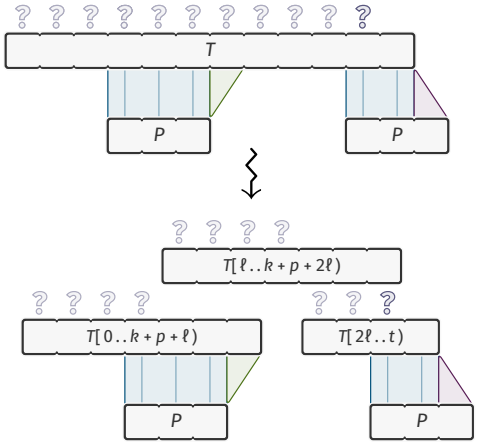
$\rightsquigarrow$ Useful special cases: $\ell = k$ and $\ell = 0.5\,p$



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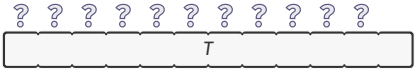

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"Filter and Verify Paradigm"

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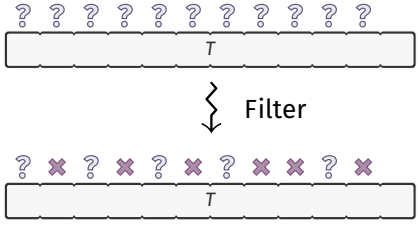

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Step 1, Filter: (typically fast)
Compute (small) superset of starting positions

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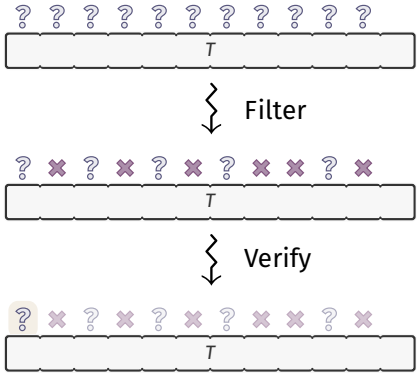
Check for occ at each remaining position



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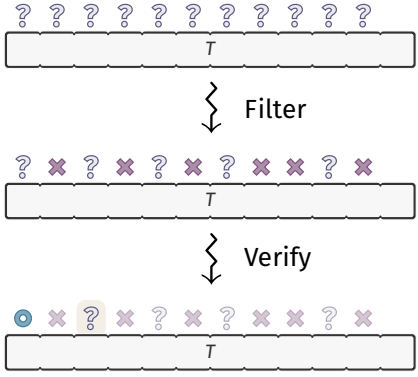

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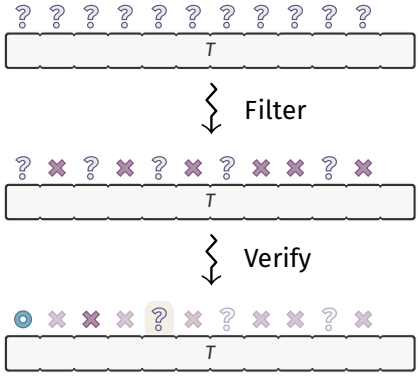

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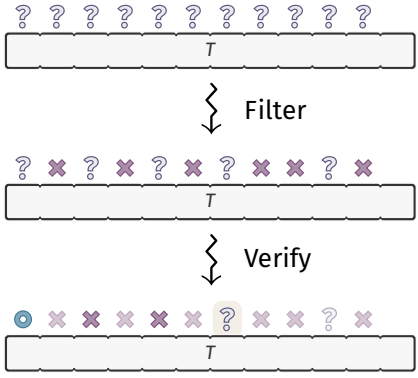

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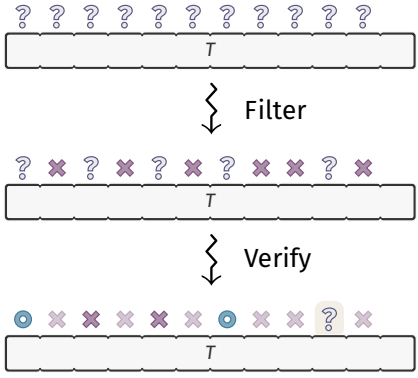

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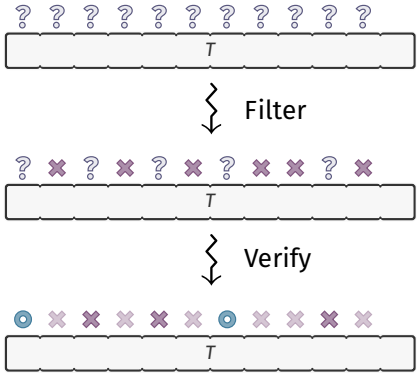

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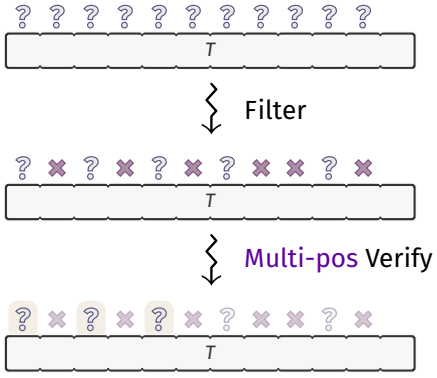
Check for occ at each remaining position, multiple positions at once



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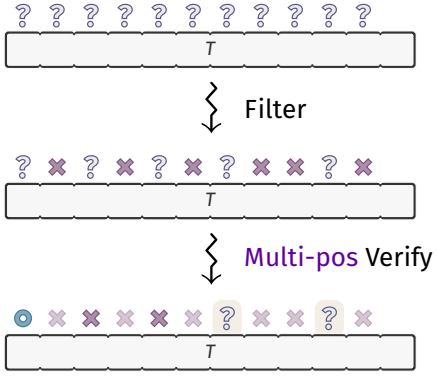

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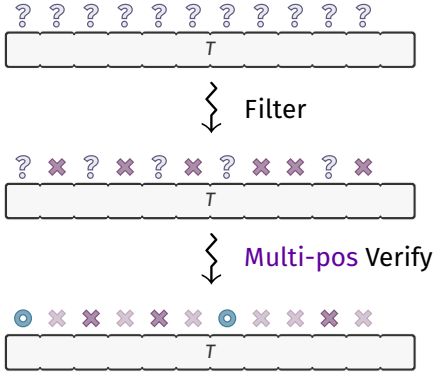

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Given: text T, pattern P, integer k; Find: (starting pos. of) substrings of T at edit distance $\leq k$ to P.

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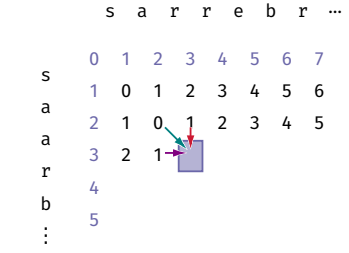
Textbook DP $\rightsquigarrow O(tp)$ [Sellers, 1980] and others	Verify pos 1 by 1 $\rightsquigarrow O(t^2 p)$	substr



$e_{i,0} = i$ (del $T[0..i)$); $e_{0,j} = j$ (ins $P[0..j)$)

$$e_{i,j} = \min \begin{cases} e_{i-1,j} + 1 & \text{(del from } T\text{)} \\ e_{i,j-1} + 1 & \text{(ins in } T\text{)} \\ e_{i-1,j-1} + (T[i] \neq P[j]) & \text{(match/subst)} \end{cases}$$

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[Sellers, 1980] and others	Verify all pos at once $\rightsquigarrow O(tp)$	start pos

		r	b	e	r	r	a	s	…
	0	0	0	0	0	0	0	0	
b	1	1	0	1	1	1	1	1	
r	2	1	1	1	1	1	2	2	
a	3	2	2	2	2	2	1	2	
a	4	3	3	3	3	3	2	2	
s	5	4	4	4	4	4	3	2	

$e_{i,0} = 0$ (del $T[0..i)$); $e_{0,j} = j$ (ins $P[0..j)$)

$$e_{i,j} = \min \begin{cases} e_{i-1,j} + 1 & \text{(del from } T\text{)} \\ e_{i,j-1} + 1 & \text{(ins in } T\text{)} \\ e_{i-1,j-1} + (T[i] \neq P[j]) & \text{(match/subst)} \end{cases}$$

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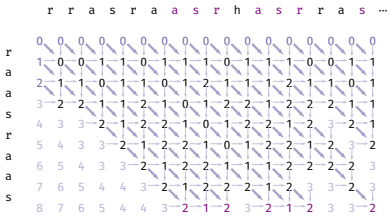
DP, diagonally compute only entries $\leq k$ $\rightsquigarrow O(t + kp)$	Verify pos 1 by 1 $\rightsquigarrow O(tkp)$ Verify $\ell \leq t$ pos at once $\rightsquigarrow O(t/\ell \cdot (\ell + k)p) = O(tp)$	substr start pos



Prune whenever $e_{i,j} > k$.

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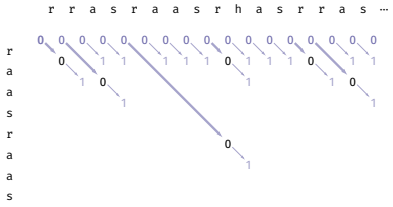
compute DP diagonals, skip over common substr in $O(1)$ $\rightsquigarrow O(t + k^2)$ [Landau, Vishkin'89].	Verify pos 1 by 1 $\rightsquigarrow O(tk^2)$ Verify $\ell \leq t$ pos at once $\rightsquigarrow O(t/\ell \cdot (\ell + k)k) = O(tk)$	substr start pos



Comp. furthest pos on diag d reachable w/ $\ell = 0, \ldots, k$ edits.
$\rightsquigarrow$ Jump over equal substrings in $O(1)$ time

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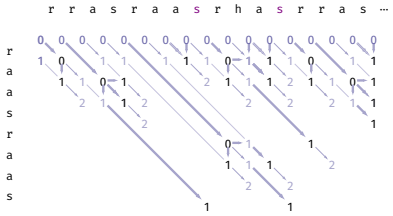

Comp. furthest pos on diag d reachable w/ $\ell = 0, \ldots, k$ edits.
$\rightsquigarrow$ Jump over equal substrings in $O(1)$ time

early 1980's

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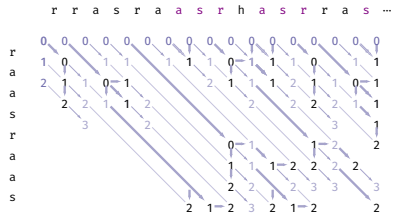

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$\rightsquigarrow$ Didn't use filter yet...

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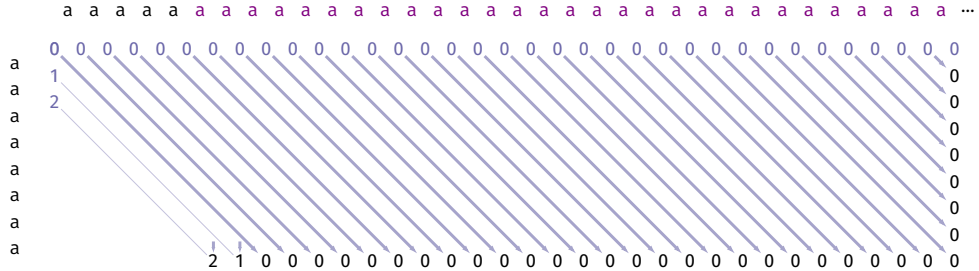

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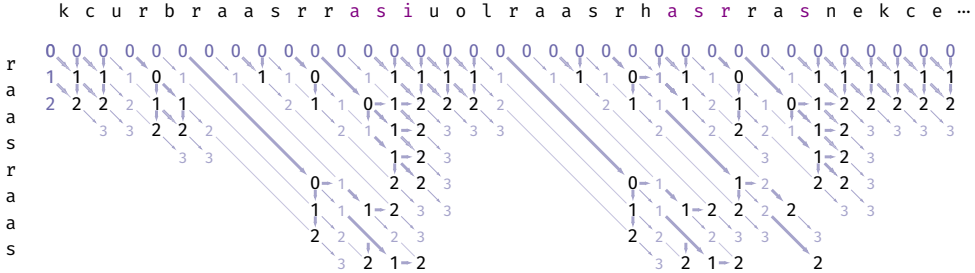

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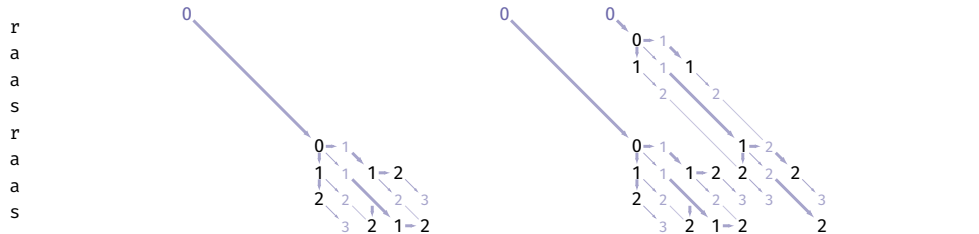

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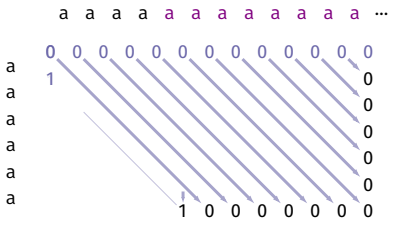

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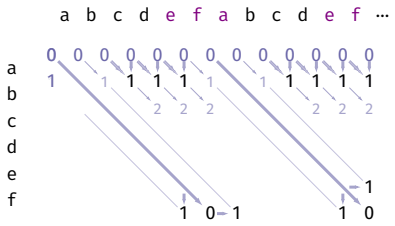

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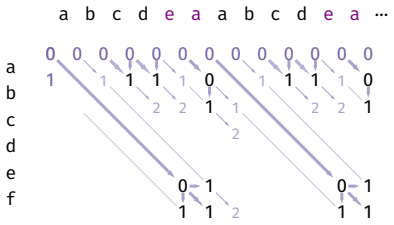

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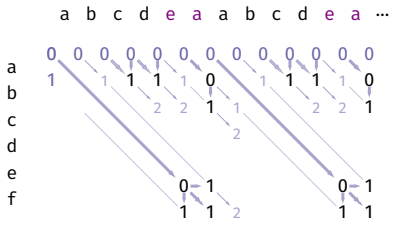
  - ⇝ Use $> k$ disjoint breaks



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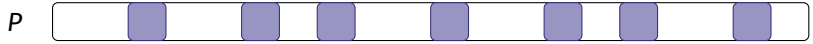

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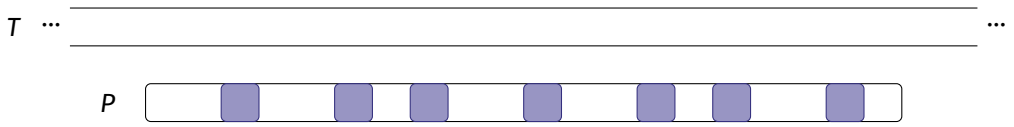

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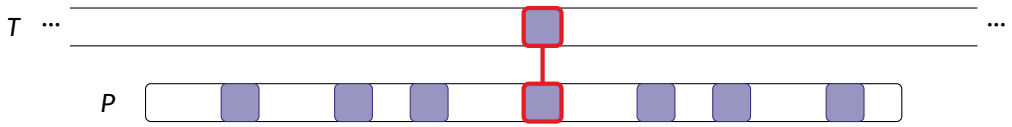

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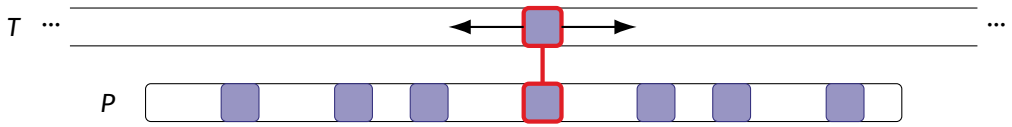

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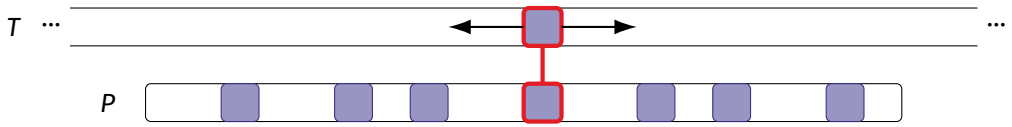

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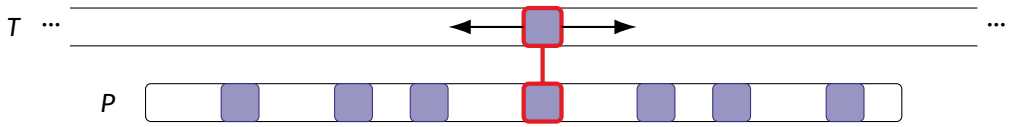

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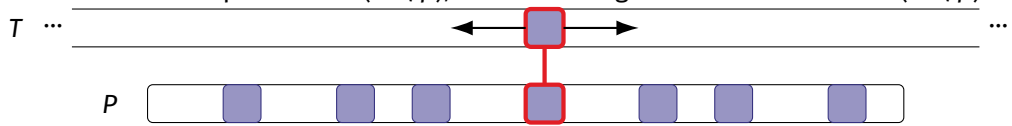
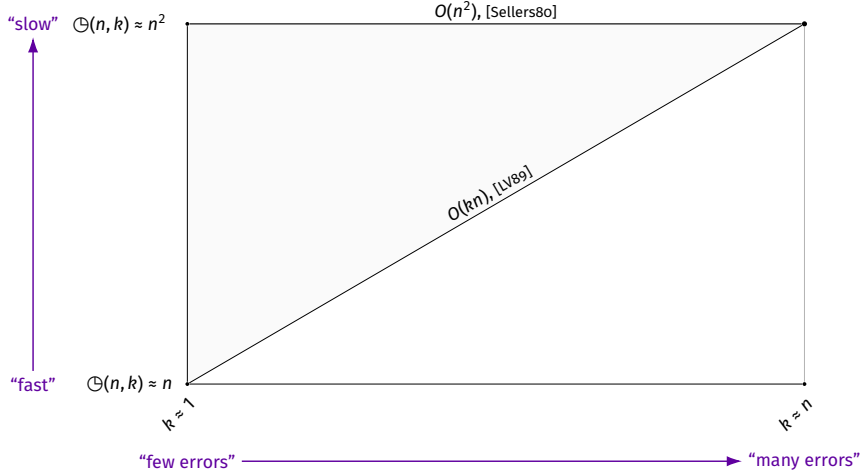


Recap: State-of-the-Art Algorithms



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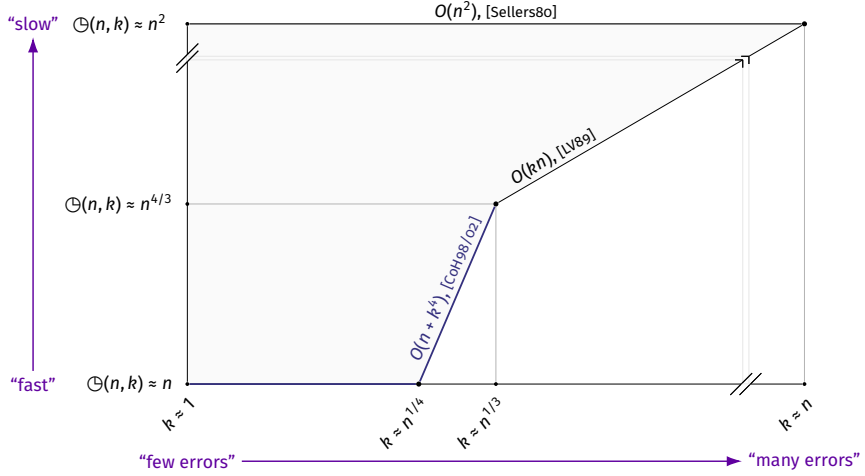

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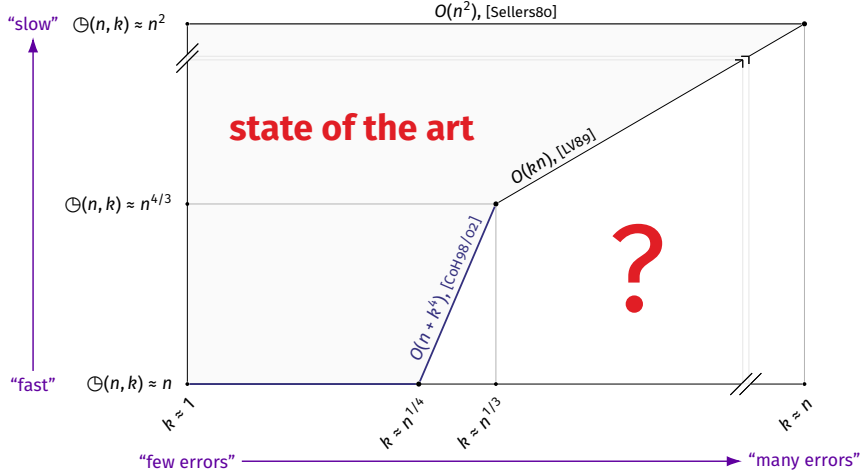

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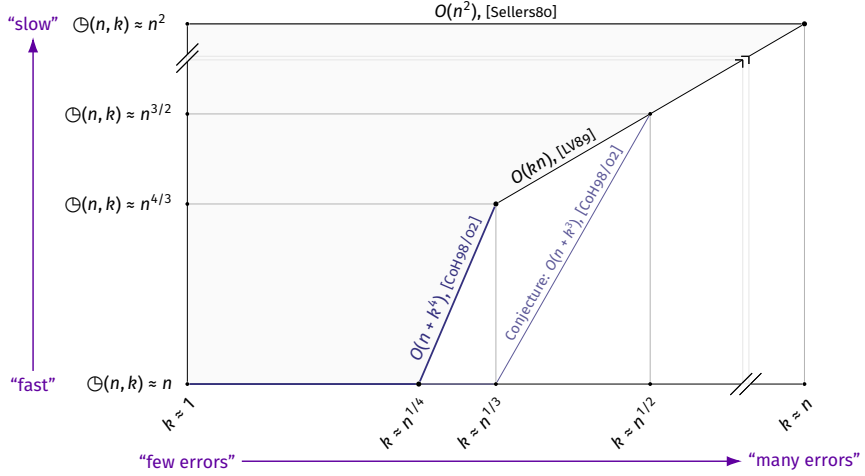

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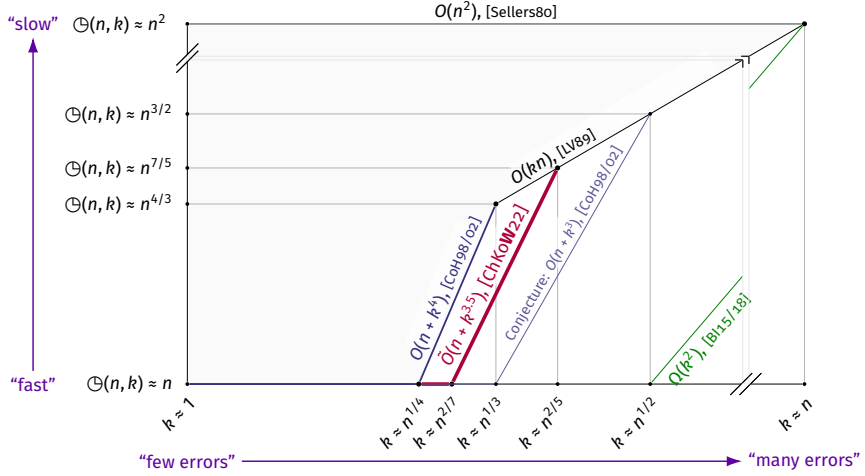

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Why did a new improvement take 24 years?

Detour: Why did a new improvement for Approximate String Matching take 24 years?

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$\rightsquigarrow$ Yields lower bound of $O(t + k^2 \cdot t/p)$ for Approximate String Matching.

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$\rightsquigarrow$ String Matching with Mismatches was "fully" understood only in 2017.

How do we obtain faster algorithms?

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New insights into the solution structure of
Approximate String Matching

Think: Better filter + more structure when filter fails

Step 0:

What is the solution structure of
Exact String Matching?

(The) Period of a String

$p > 0$ is a period of a string S if $S[\,i\,] = S[\,i + p\,]$ for all $i = 1, \ldots, |S| - p$.

The period of S: smallest period of S, write $\mathrm{per}(S)$.

S is periodic: $\mathrm{per}(S) \leq |S|/2$

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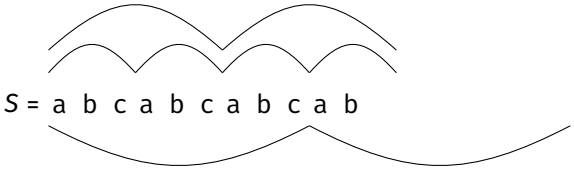

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6 and 9 also periods of S

"Periodicity Lemma"

(Folklore)

Text T, pattern P with $t \leq \frac{3}{2}p$; one of the following holds

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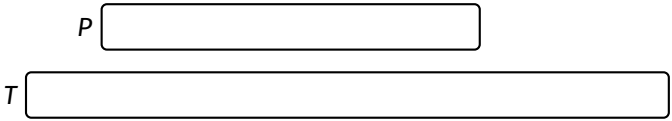

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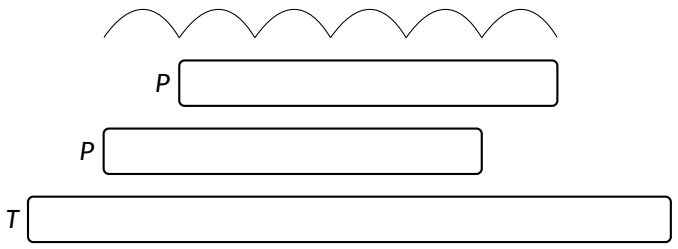

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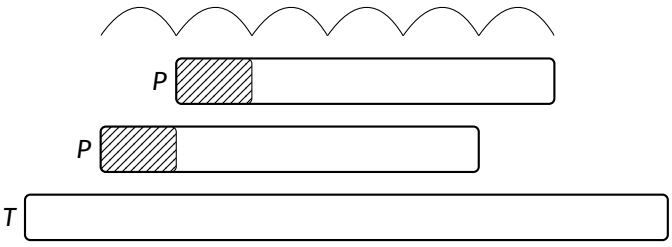

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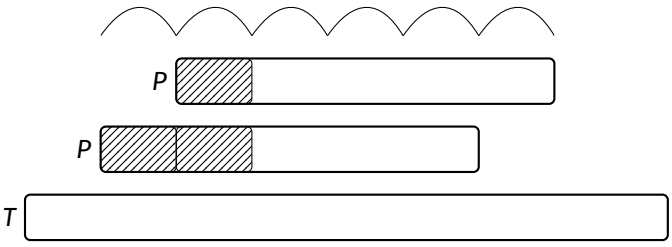

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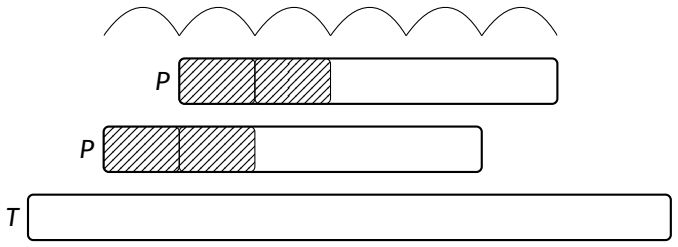

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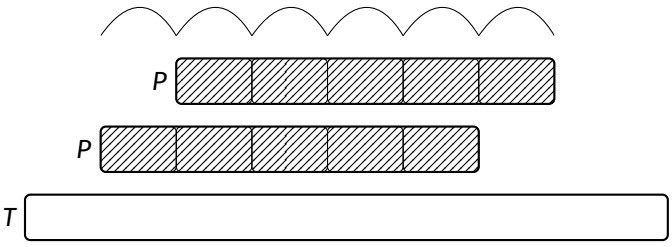
- P (and the relevant part of T) are periodic with some period Q.



"Periodicity Lemma"

(Folklore)

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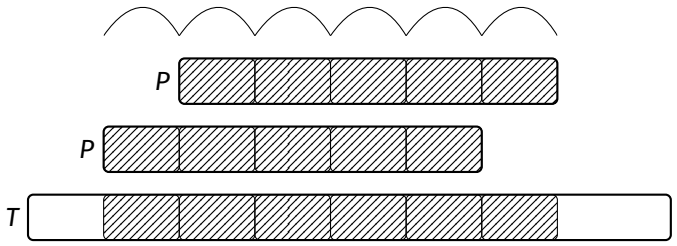

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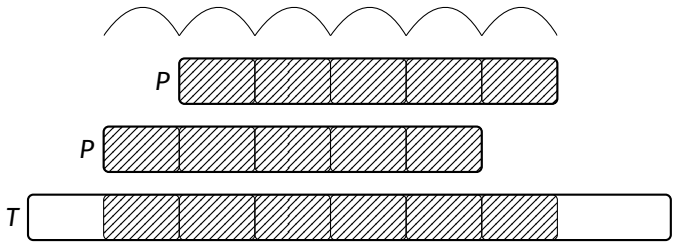


Standard Trick is useful here:

For $t \gg p$ consider separately $O(t/p)$ fragments of T; then Periodicity Lemma applies

Step 1:

What is the solution structure of
String Matching with Mismatches?

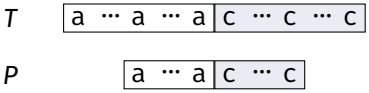
Periodicity Lemma (Folklore)

Text T, pattern P with $t \leq {}^3/_2\, p$; P appears ≤ 1 times in T or P is periodic with some period Q.

Main Result (Mismatches) [BKü**W**'19; ChKo**W**'20]

Text T, pattern P with $t \leq {}^3/_2\, p$; threshold k, one of the following holds

P appears $\leq O(k)$ times in T



a···ac···c appears $2k$ times

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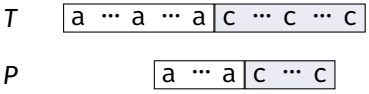

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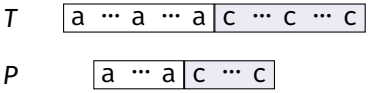

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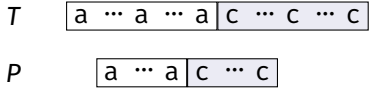
Text T, pattern P with $t \leq {}^3/_2\, p$; threshold k, one of the following holds

P appears $\leq O(k)$ times in T or P is almost periodic with some period Q



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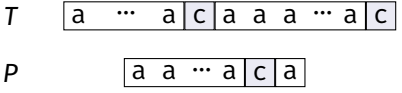

aa···aca has approximate period a
(is at HD $\leq 2k$ from Q^∞)

Step 1.5:

The solution structure of
Approximate String Matching.

Main Result (Mismatches) [BKüW'19; ChKoW'20]

T, P with $t \leq 3/2\, p$; threshold k, P appears $O(k)$ times in T or P is almost periodic with some period Q

Main Result (Edits) [ChKoW'20]

Text T, pattern P with $t \leq 3/2\, p$; threshold k, one of the following holds

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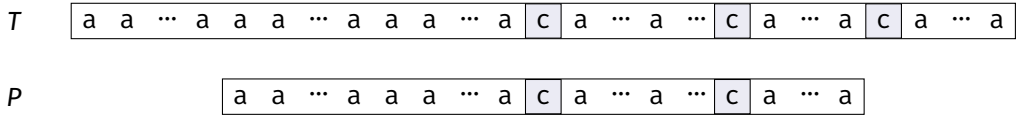

aa···aaa···aca···a···ca···a appears $\approx k^2$ times

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Key Intermediate Result (Analyze) [ChKoW'20]

Every string P satisfies at least one of

- P is almost periodic. aaacaaaaaaaaacaaaaaaa

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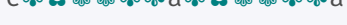

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Every string P satisfies at least one of

- P has $2k$ disjoint, long breaks — c❀✿❁❁❀❀a❀✿❁❁❀❀a
- P has disjoint repetitive regions that cover ${}^{3}/_{8}\, P$ — ❀❀aaaaaa❀❀ccaccc❀❀
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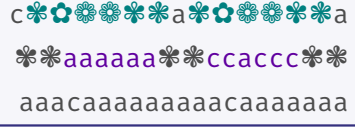

Analyze implies Main Result:
Almost periodicity $\rightsquigarrow$ as in Main Result
Breaks, repetitive regions $\rightsquigarrow$ good filter (as in [Cole, Hariharan'98])

Key Intermediate Result (Analyze)

[ChKoW'20]

Every string P satisfies at least one of

- P has $2k$ disjoint, long breaks — c❀✿❁❁❀❀a❀✿❁❁❀❀a
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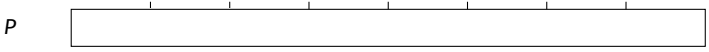

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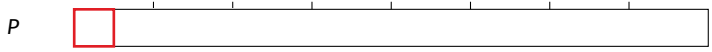

- Process P from left to right, $p/8k$ new characters at a time.

Key Intermediate Result (Analyze)

[ChKoW'20]

Every string P satisfies at least one of

- P has $2k$ disjoint, long breaks — c❃✿❁❁❃❃a❃✿❁❁❃❃a
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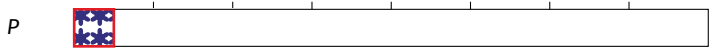

Breaks B = {  }

Repetitive Regions R = { }

- If a fragment is a break, add it to the found breaks.

Key Intermediate Result (Analyze)

[ChKoW'20]

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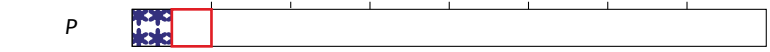

Breaks B = {  }

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- Otherwise, find the shortest prefix (longer than $p/8k$) that is a repetitive region.

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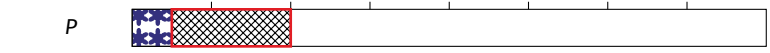

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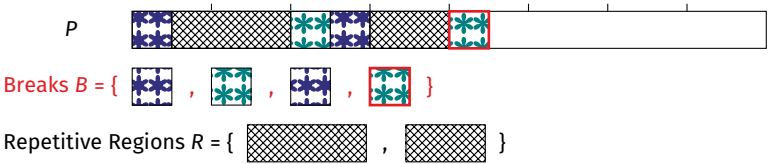

- If we found $2k$ breaks, return the breaks.

Key Intermediate Result (Analyze)

[ChKoW'20]

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- P has $2k$ disjoint, long breaks — c❀✿❁❁❀❀a❀✿❁❁❀❀a
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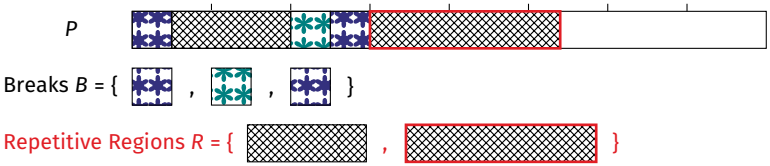

- If the total length of the repetitive regions is $> 3/8 \cdot p$, return the repetitive regions.

Key Intermediate Result (Analyze)

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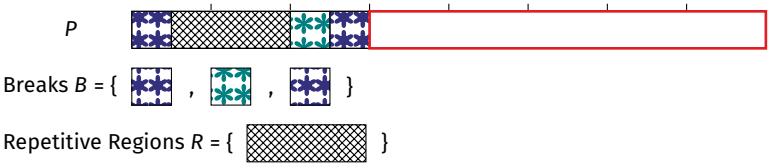

- If we reach the end of P, try to find a single repetitive region starting from the end.

Key Intermediate Result (Analyze)

[ChKoW'20]

Every string P satisfies at least one of

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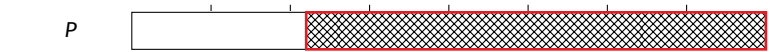

Breaks B = { }

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- If we found a repetitive region, return it.

Key Intermediate Result (Analyze)

[ChKoW'20]

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- P is almost periodic. — aaacaaaaaaaaacaaaaaaa

Breaks B = { }

Repetitive Regions R = { }

- If we again don't obtain a repetitive region, P is almost periodic.

Key Intermediate Result (Analyze) ✓

[ChKoW'20]

Every string P satisfies at least one of

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How do we turn our insights
into faster algorithms?

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Need to tackle three cases.

- P contains $2k$ disjoint breaks;
- P contains disjoint repetitive regions R_i;
- P is almost periodic

How do we turn our insights into faster algorithms?

Need to tackle ~~three~~ two cases.

- P contains $2k$ disjoint breaks;
- ~~P contains disjoint repetitive regions R_i;~~ $\rightsquigarrow$ Follows from the other two cases.
- P is almost periodic

How do we turn our insights into faster algorithms?

Need to tackle ~~three~~ one case.

- ~~P contains $2k$ disjoint breaks;~~ $\rightsquigarrow$ Adaption of [Cole,Hariharan'98] yields $O(|T| + |T|/|P| \cdot k^3)$.
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[ChKo**W**'22]

Reduction to Periodic Patterns

Algorithm for almost periodic case in time $\tilde{O}(|T| + k^a \cdot |T|/|P|)$, for $a \geq 3$
$\Longrightarrow$ Algorithm for general case in time $\tilde{O}(|T| + k^a \cdot |T|/|P|)$

[ChKo**W**'22]

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Need to tackle: P is almost periodic

Reduction to Periodic Patterns [ChKo**W**'22]

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Need to tackle: P is almost periodic

- In [ChKo**W**'20], use elaborate marking scheme to obtain $O(|T| + |T|/|P| \cdot k^4)$ algorithm
  $\rightsquigarrow$ Not faster than [Cole,Hariharan'98]

[ChKo**W**'22]

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  - Refinement of algorithm from [ChKo**W**'20]
  - New algorithm based on "Seaweed Technology" of [Tiskin'10,'15]

  $\rightsquigarrow$ Yields $O(|T| + |T|/|P| \cdot k^{3.5})$ algorithm

Reduction to Periodic Patterns

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    New in [ChKo**W**'25]: Simpler plug-in replacement for "Seaweed Technology" based on SMAWK (efficient $(\min, +)$-multiplication of Monge matrices)

  $\rightsquigarrow$ Yields $O(|T| + |T|/|P| \cdot k^{3.5})$ algorithm

Beyond Strings: The PILLAR Model

- All of our algorithms rely on a small set of essential operations:
  - $\texttt{LCP}(S, T)$: Compute the length of the longest common prefix of S and T.
  - $\texttt{LCP}^R(S, T)$: Compute the length of the longest common suffix of S and T.
  - $\texttt{IPM}(P, T)$: Compute all exact matches of P in T.
  - $\texttt{Length}(S)$: Compute the length s of S.
  - $\texttt{Access}(S, i)$: Retrieve the character $S[i]$.
  - $\texttt{Extract}(S, \ell, r)$: Extract the fragment (or substring) $S[\ell..r]$ from S.

Main Result (PILLAR Algorithm) [ChKoW'22]

For $t \leq 3/2\, p + k$; ASM can be solved using $O(k^{3.5})$ PILLAR operations

Beyond Strings: The PILLAR Model

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  - LCP(S, T): Compute the length of the longest common prefix of S and T.
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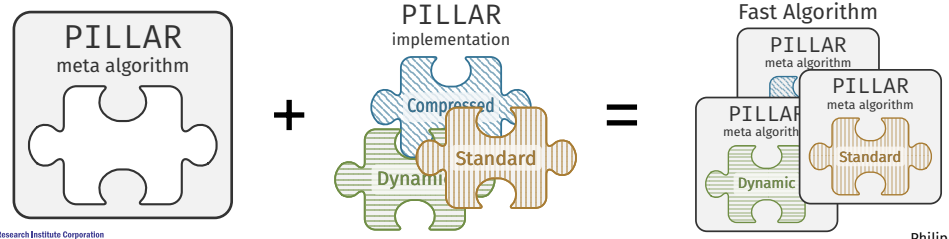

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- Bottom-line, useful design principle: meta algorithm (PILLAR) + simple-to-solve subproblems

Spotlight Extension: Quantum Algorithms for ASM

Approximate String Matching

early 1980's

Given: text T, pattern P, integer k; Find: (starting pos. of) substrings of T at edit distance $\leq k$ to P.

- P/T given as oracle $\rightsquigarrow$ queries possible in superposition
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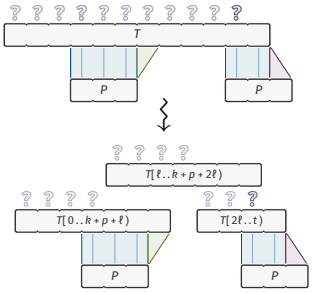
Toy example: "Standard Trick" for ASM decision version (∃? occ)
Split T into overlapping fragments of len $\ell + p + k$
$\rightsquigarrow O(t/\ell)$ instances, each "responsible" for its first ℓ positions
$\rightsquigarrow$ Classically: $O(t/\ell)$ time overhead (evaluate each inst. 1 by 1)



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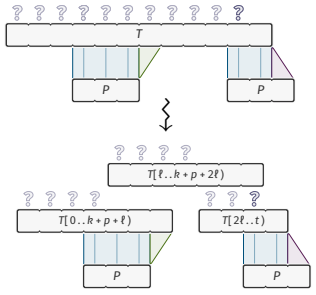
Quantum setting: Use amplitude amp. / Grover's algo.
$\tilde{O}(\sqrt{t/\ell})$ time and $O(\sqrt{t/\ell})$ evaluations of single inst.



Amplitude Amplification

[Gro96, BHMT02]

Given function $f : [0..n) \to \{0, 1\}$
Can (wp ≥ 2/3) obtain $x \in f^{-1}(1)$ in $\tilde{O}(\sqrt{n})$ time + $O(\sqrt{n})$ queries to f

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For length-n strings,
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Yields $\tilde{O}(t/p \cdot k^{3.5} \cdot \sqrt{p})$ quantum time algo for ASM; ($\tilde{O}(\sqrt{t/p} \cdot k^{3.5} \cdot \sqrt{p})$ for $\exists$?) (algos are correct whp.)

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Are these algorithms optimal? $\rightsquigarrow$ No.

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  $\rightsquigarrow$ Can use `PILLAR` algorithm for fully-compressed setting on lossy compression!

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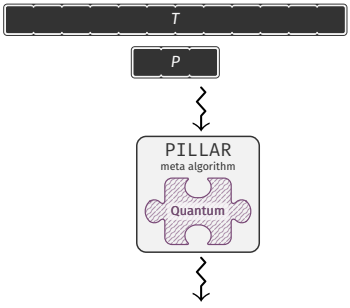

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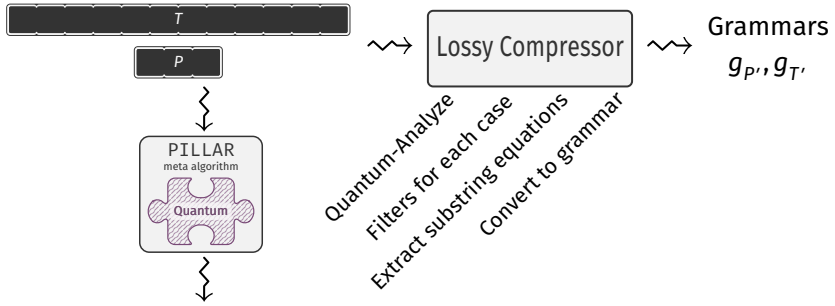

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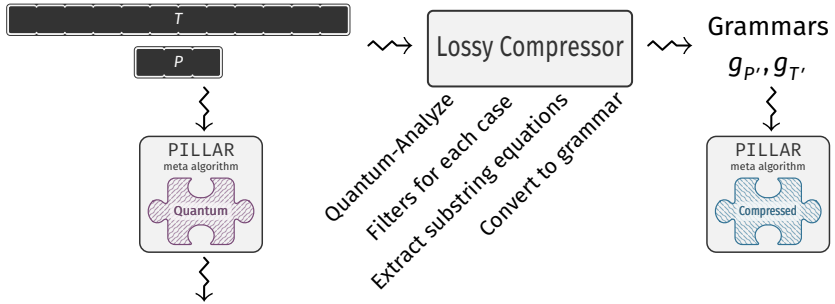

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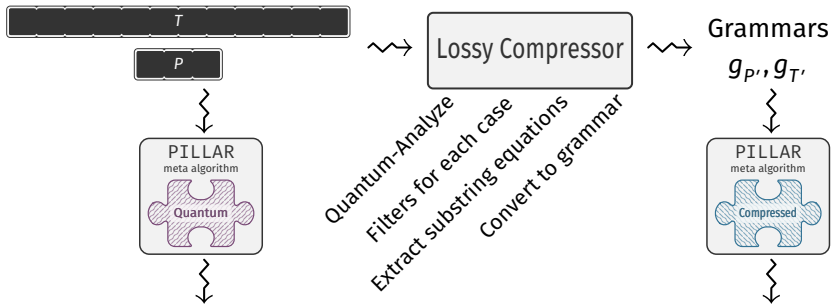

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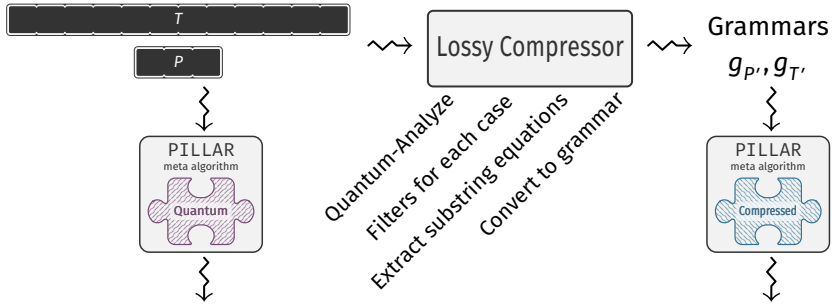

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$\rightsquigarrow$ Queries are (near) optimal [KoN**W**'24], time overhead from PILLAR (but future-proof)

Spotlight Extension: String Matching with Weighted Edits

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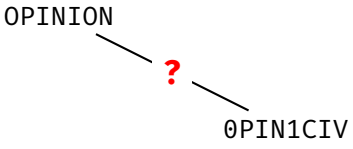

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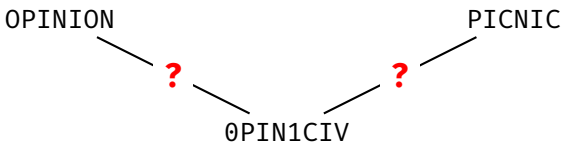

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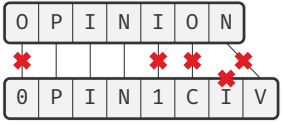

$\mathrm{ED}(\texttt{0PIN1CIV}, \texttt{OPINION}) = 5$

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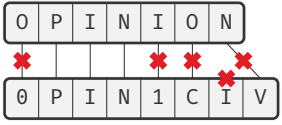

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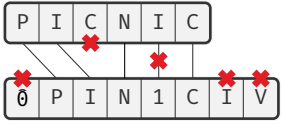

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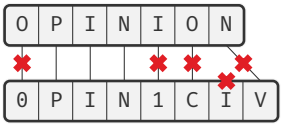
Weighted Edit Distance

$\mathrm{ED}^w(X, Y)$

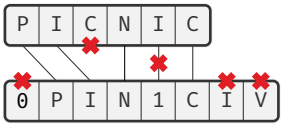
Min cost of transforming X to Y using character edits, where:

- inserting y costs $w(\varepsilon, y)$;
- deleting x costs $w(x, \varepsilon)$;
- substituting x for y costs $w(x, y)$.

$w(0, 0) := 1 \quad w(1, \mathrm{I}) := 1 \quad w(\mathrm{C}, 0) := 1$



$w(*, *) := 2 \quad w(*, \varepsilon) := 1 \quad w(\varepsilon, *) := 10$



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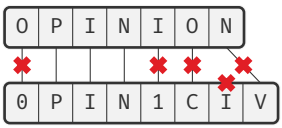

$\mathrm{ED}^w(\mathtt{0PIN1CIV}, \mathtt{OPINION}) = 6$

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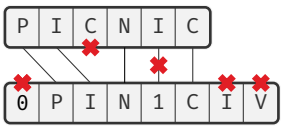

$\mathrm{ED}^w(\mathtt{0PIN1CIV}, \mathtt{PICNIC}) \leq 14$

$\mathrm{ED}^w(X, Y)$

Weighted Edit Distance

Min cost of transforming X to Y using character edits, where:

- inserting y costs $w(\varepsilon, y)$;
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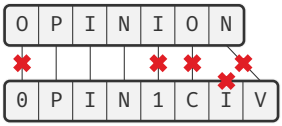
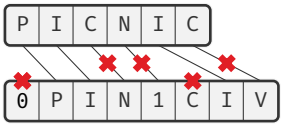
$w(0, \mathtt{O}) := 1 \quad w(1, \mathtt{I}) := 1 \quad w(\mathtt{C}, \mathtt{O}) := 1$



$\mathrm{ED}^w(\mathtt{0PIN1CIV}, \mathtt{OPINION}) = 6$

$w(*, *) := 2 \quad w(*, \varepsilon) := 1 \quad w(\varepsilon, *) := 10$



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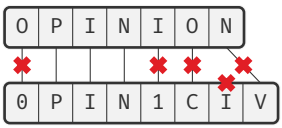

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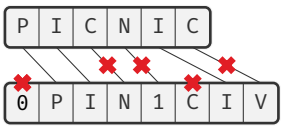

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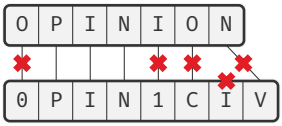

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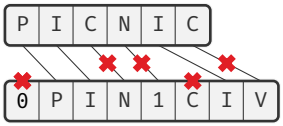

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Otherwise: could scale weights and condition $\mathrm{ED}^w_{\leq k}(X, Y) \leq k$ (e.g. in ASM) becomes meaningless.

Recall crucial subroutine for ASM:

Verify that occurrence starts at given (interval of) position of T

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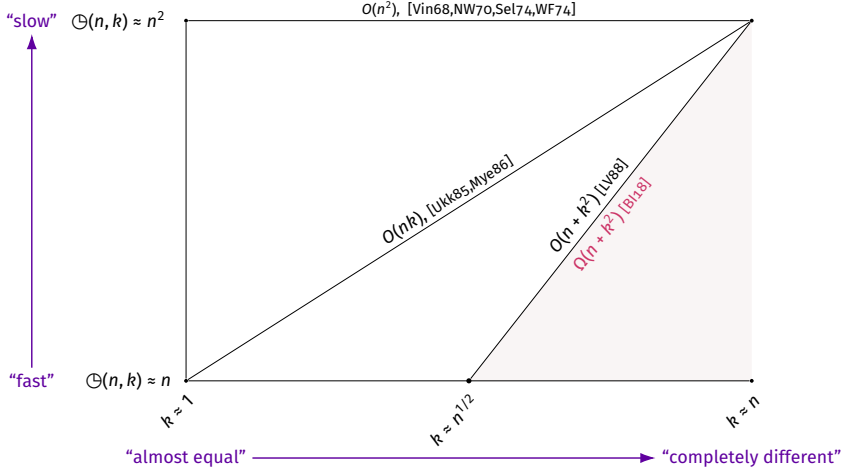
$\rightsquigarrow$ Need to compute Bounded Weighted Edit Distance

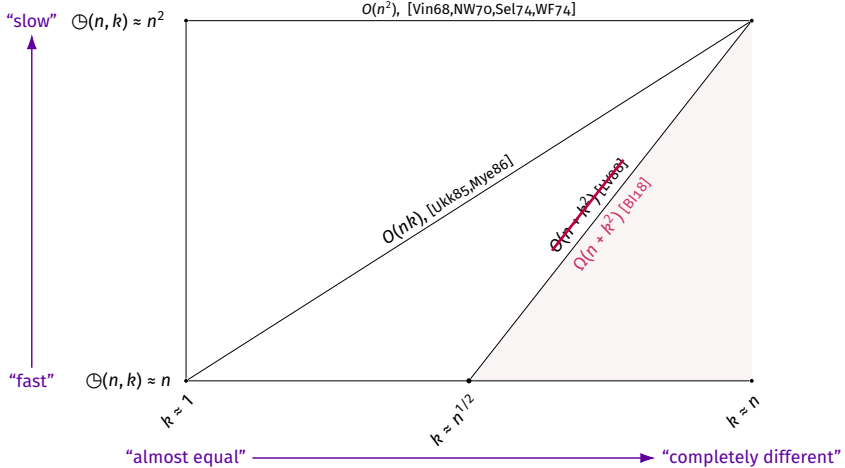
Recall: Algorithms for (Bounded) Edit Distance / Verify



Existing algorithms for Edit Distance ED(X, Y), where $|X|, |Y| \leq n$

What about Weighted Edit Distance?

Algorithms for (Bounded) Weighted Edit Distance / Weighted Verify



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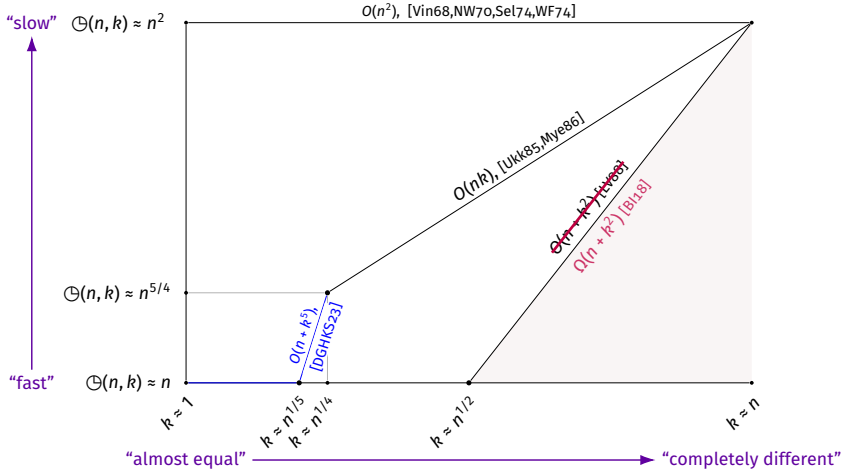

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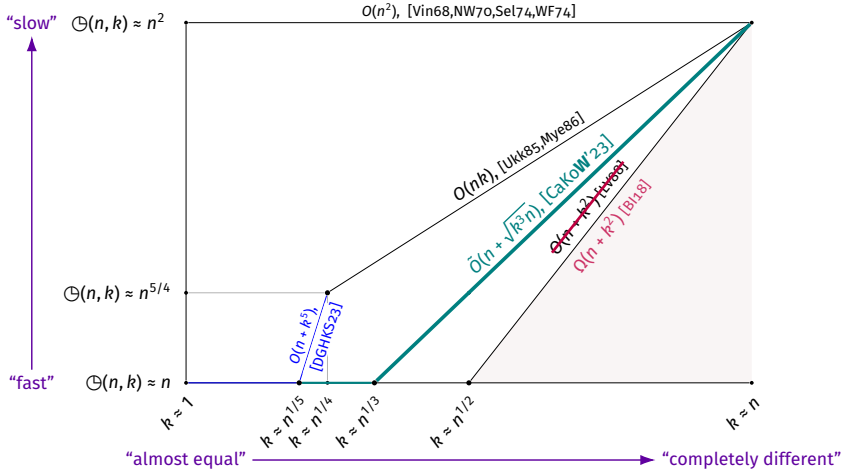

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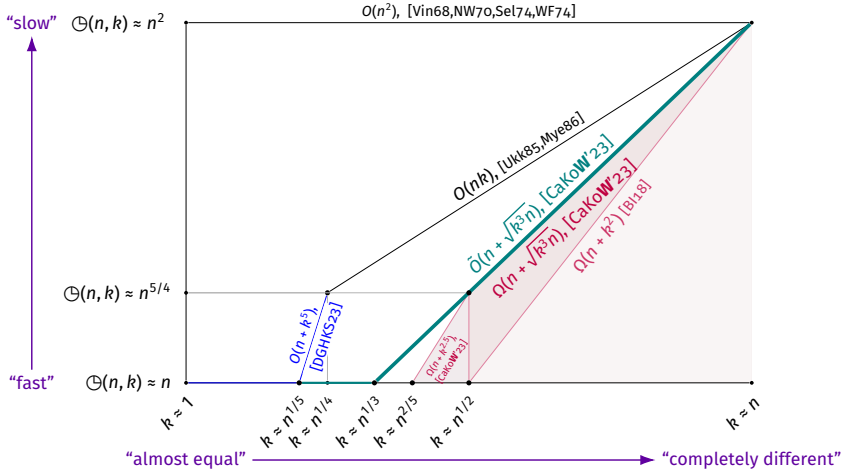

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Strings X, Y each of length at most n

Oracle access to (normalized) weight function w

Can compute $k = \mathrm{ED}^w(X, Y)$ in time $O(n + \sqrt{nk^3} \log^3 n)$

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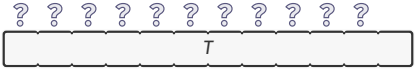
$\rightsquigarrow$ What about $\tilde{O}(kt)$-time for weighted ASM?
With Standard Trick, UB yields $\tilde{O}(t/k \cdot \sqrt{pk^3}) = \tilde{O}(t\sqrt{pk})$
LB rules out $\tilde{O}(kt)$ via better WED algo $\rightsquigarrow$ need different approach

Weighted Approximate String Matching

Given: T, P, k, oracle to w; Find: (starting pos. of) substr. of T at weighted ED $\leq k$ to P.

Focus: Obtain starting positions of occurrences



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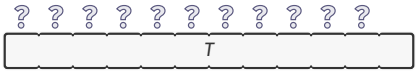

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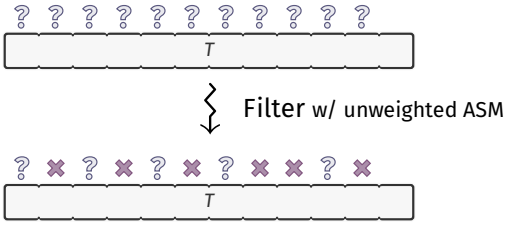

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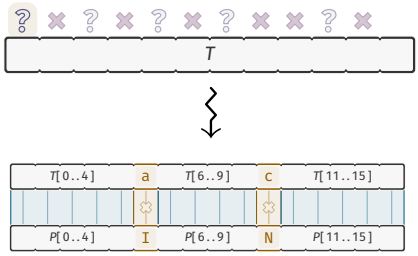

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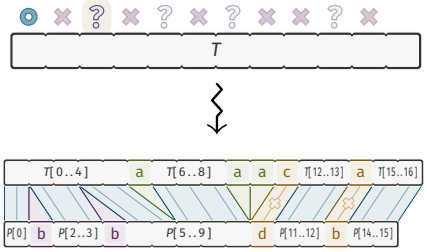

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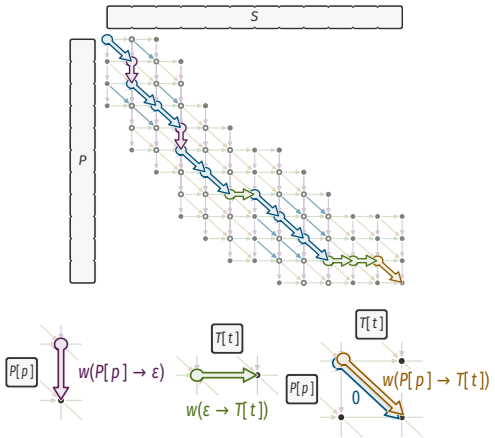

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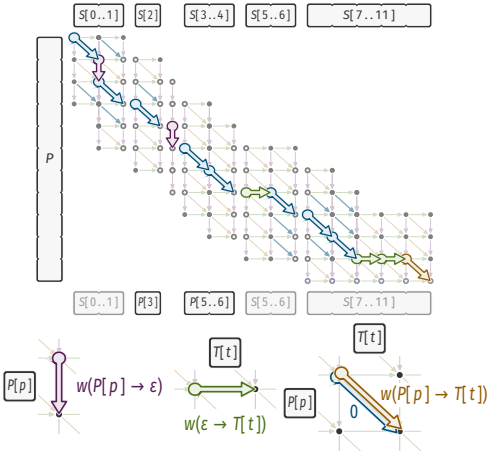

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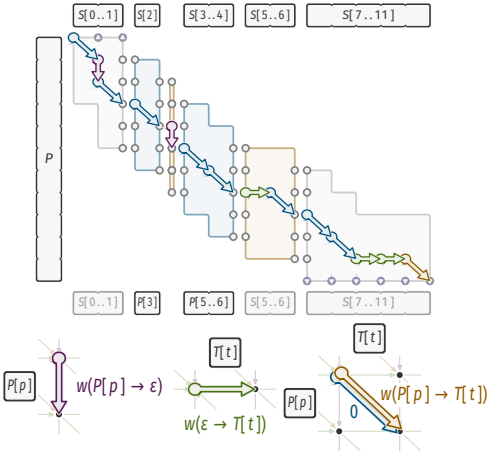

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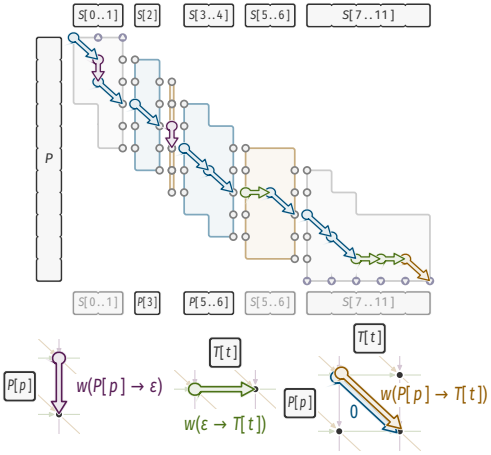
$\rightsquigarrow$ Verify $\leq k$ pos in $\tilde{O}(k^2)$ after $\tilde{O}(kp)$ prep. $\rightsquigarrow$ $\tilde{O}(kt)$ total



Weighted ASM: Bottom Lines

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But, also interesting non-PILLAR algorithms out there.

Open Problems and Future Directions

- Big Question: Close gap between UB $O(n + k^{3.5})$ / LB $\Omega(n + k^2)$ for ASM.
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Start
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