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Hideo KANEMITSU, Hideaki IMAI, Masaaki MIYASKOSHI

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Definitions and Properties of (Local) Minima and Multimodal Functions using Level Set for Continuous Optimization Problems

Hideo KANEMITSU†, Hideaki IMAI‡, Masaaki MIYASKOSHI‡,

†Hakodate Campus, Hokkaido University of Education, Hakodate 040-8567, Japan.
‡Graduate School of Information Science and Technology Hokkaido University
Email: † kanemitsu.hideo@h.hokkyodai.ac.jp, ‡ {imai,miyakosi}@ist.hokudai.ac.jp

Abstract—We show mathematical structures on optimal solutions in a continuous optimization problem with a continuous multivariate multimodal objective function : "minimize : $f(x)$ subject to $x \in S \subset \mathbb{R}^n$. For realizing the purpose, we show definitions of (local) minima using neighborhood of each minimum and using some types of level set. Moreover, we describe relationship among the previous definitions and new definitions of (local) minima, differences of concept of solutions, and the number of solutions especially in case where there exists flat regions on a function. The new definitions in this paper is simpler than the previous definitions of (local) minima.

1. Introduction

Continuous optimization problem:"Minimize (min.) a objective function $f(\boldsymbol{x}) \equiv f(x_1, x_2, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$ of n-variable; subject to (s.t.) $\boldsymbol{x} \in S$ (compact set)" has been applied to many fields. Especially, in case where objective functions (generalized) convex function, many theoretical results have been obtained[1, 6].

On the other hand, many studies have been done for problems such that objective functions are nonconvex or multimodal functions. However, many of these studies are proposing algorithms or investigating performance and behavior of algorithm's, theoretical studies for these problems insufficient than for (generalized) convex problems. For examples, It has not even been defined number of modality for multimodal problems. Demidenko[2] was investigate basic properties for nonconvex or multimodal problem. However, it is not take no account of existence of flat regions in problem.

In this paper, we proposed the local minimal values set (l.m.v.s.) as a new concept of optimal solutions, and defined the number of modality as the number of connected components.

The remainder of the paper is organized as follows. Problem in our paper, definitions of previous local minima and level sets are given in Sect. 2. In Sect. 3, four types of sets of local minima are formulated, and mainly inclusion relations are shown. Definitions and inclusion relations by level sets are shown in Sect. 4. At last, concluding remarks are described in Sect. 5.

2. Preliminaries

2.1. Optimization Problem and its assumptions

Continuous optimization problems is formulated as follows:

$$(\mathrm{P}) \begin{cases} \text{min.} & \tilde{f}(\boldsymbol{x}) \equiv \tilde{f}(x_1, x_2, \ldots, x_n), \\ \text{s.t.} & \boldsymbol{x} \in S \subset \mathbb{R}^n. \end{cases}$$

For the problem, we have the following assumptions.

Assumption 1 *(A1) $S \subset \mathbb{R}^n$ is compact, and (A2) function f is lower semi continuous.*

For studying the problem (P) as a unconstrained problem on $\mathbb{R}^n$, we define a following objective function f.

Definition 2 *An extend function $f : \mathbb{R}^n \to (-\infty, +\infty]$ is defined as follows.*

$$f(\boldsymbol{x}) = \begin{cases} \tilde{f}(\boldsymbol{x}), & \boldsymbol{x} \in S\,; \\ +\infty, & \boldsymbol{x} \notin S\,. \end{cases} \tag{1}$$

2.2. Definitions of previous and these inclusion relations

At first, we define (global) minimum.

Definition 3 *A minimum $\boldsymbol{x}^{**}$ is*

$$\exists \boldsymbol{x}^{**} \in S,\ \forall \boldsymbol{x} \in \mathbb{R}^n; \quad f(\boldsymbol{x}^{**}) \leq f(\boldsymbol{x}), \tag{2}$$

and set of these points is denoted by $\boldsymbol{X}^{**}$.

Next, we show the previous definitions on local minima.

Definition 4 *For a point $\boldsymbol{x}^*$ and a ball $B(\boldsymbol{x}^*, \delta_1)$ with the centered point $\boldsymbol{x}^*$ and a radius δ_1, if the following equation:*

$$\exists \boldsymbol{x}^*,\ \exists \delta_1 > 0,\ \forall \boldsymbol{x} \in B(\boldsymbol{x}^*, \delta_1); \quad f(\boldsymbol{x}^*) \leq f(\boldsymbol{x}) \tag{3}$$

holds, then the point $\boldsymbol{x}^*$ is called the local minimum. *Moreover if the above equation is satisfied for a point $\boldsymbol{x}^*$ in $-f$, then the point is called the* local maximum.

Definition 5 *For a point $\boldsymbol{x}_s^*$ and a ball $B(\boldsymbol{x}_s^*, \delta_2)$ with the centered point $\boldsymbol{x}_s^*$ and a radius δ_2, if the following equation:*

$$\exists \boldsymbol{x}_s^*,\ \exists \delta_2 > 0,\ \forall \boldsymbol{x} \in B(\boldsymbol{x}_s^*, \delta_2) \setminus \boldsymbol{x}_s^*; \quad f(\boldsymbol{x}_s^*) \leq f(\boldsymbol{x}) \tag{4}$$

holds, then the point $\boldsymbol{x}_s^*$ is called the strictly local minimum.

Definition 6 *For a point $\boldsymbol{x}_i^*$ and a ball $B(\boldsymbol{x}_i^*, \delta_3)$ with the centered point $\boldsymbol{x}_i^*$ and a radius δ_3, if the following equation:*

$$\exists \boldsymbol{x}_i^*,\ \exists \delta_3 > 0,\ \forall \boldsymbol{x} \in B(\boldsymbol{x}_i^*, \delta_3) \setminus \boldsymbol{x}_i^*; \quad f(\boldsymbol{x}_i^*) \leq f(\boldsymbol{x}) \tag{5}$$

holds, then the point $\boldsymbol{x}_i^*$ *is called the* isolated local minimum.

2.3. Definitions of level set and connected level set etc.

In this section, we define (connected) level sets at a cut end of level value of function.

Definition 7 A level set $L^{\leq}(\alpha) \subset \mathbb{R}^n$ *, a* strict level set $L^{<}(\alpha) \subset \mathbb{R}^n$ *and an* equal level set $L^{=}(\alpha) \subset \mathbb{R}^n$ *at a level* $\alpha = f(\boldsymbol{x}) \in \mathbb{R}$ *are defined respectively as follows*

$$L^{\leq}(\alpha) = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) \leq \alpha \,\}, \tag{6}$$

$$L^{<}(\alpha) = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) < \alpha \,\}, \tag{7}$$

$$L^{=}(\alpha) = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) = \alpha \,\}. \tag{8}$$

Next, we define connected level set etc.

Definition 8 *Connected level set etc. of* $\boldsymbol{x}$ ：

- Connected level set etc. ： $L_c^{\leq}(\alpha; \boldsymbol{x})$ *and* $L_c^{\leq}(f(\boldsymbol{x}))$.
  The connected component *of* $L^{\leq}(\alpha)$ *such that the component include* $\boldsymbol{x}$ *is called* connected level set*, and is denoted by* $L_c^{\leq}(\alpha; \boldsymbol{x})$*. In particular, in case where* $\alpha = f(\boldsymbol{x})$*, its set is denoted by*

$$L_c^{\leq}(f(\boldsymbol{x})) \equiv L_c^{\leq}(f(\boldsymbol{x}); \boldsymbol{x}) = L_c^{\leq}(\alpha; \boldsymbol{x}) \, (\text{at } \alpha = f(\boldsymbol{x})). \tag{9}$$

  Similarly connected equal level set *and* connected strictly level set *with level* $\alpha = f(\boldsymbol{x})$ *of* $\boldsymbol{x}$ *and can be defined, and both sets are denoted by as follows.*
- connected equal level set *with level* $f(\boldsymbol{x})$ *of* $\boldsymbol{x}$*:*

$$L_c^{=}(f(\boldsymbol{x})) \equiv L_c^{=}(f(\boldsymbol{x}); \boldsymbol{x}). \tag{10}$$

- strictly connected level set *with level* $f(\boldsymbol{x})$ *of* $\boldsymbol{x}$*:*

$$L_c^{<}(f(\boldsymbol{x})) \equiv L_c^{<}(f(\boldsymbol{x}); \boldsymbol{x}) = L_c^{\leq}(f(\boldsymbol{x})) \setminus L_c^{=}(f(\boldsymbol{x})). \tag{11}$$

3. Basic properties of local minima

3.1. Inclusion relations of three types of local minima

In this section, we show three previous types of local minimal set, and show inclusion relations of three types of local minima.

Notation 9 *The set of (minima, local minima, strictly local minima, isolated local minima) are denoted by* $\boldsymbol{X}^{**}$, $\boldsymbol{X}^*$, $\boldsymbol{X}_s^*$, $\boldsymbol{X}_i^*$*, respectively, and these sets are formulated respectively as follows.*

$$\boldsymbol{X}^{**} = \{\, \boldsymbol{x}^{**} \mid \exists \boldsymbol{x}^{**} \in S,\ \forall \boldsymbol{x} \in \mathbb{R}^n;\ f(\boldsymbol{x}^{**}) \leq f(\boldsymbol{x}) \,\}, \tag{12}$$

$$\boldsymbol{X}^{*} = \{\, \boldsymbol{x}^{*} \mid \exists \boldsymbol{x}^{*} \in S,\ \exists \delta_1 > 0,\ \forall \boldsymbol{x} \in B(\boldsymbol{x}^*, \delta_1);\ f(\boldsymbol{x}^*) \leq f(\boldsymbol{x}) \,\}, \tag{13}$$

$$\boldsymbol{X}_s^{*} = \{\, \boldsymbol{x}_s^{*} \mid \exists \boldsymbol{x}_s^{*} \in S,\ \exists \delta_2 > 0,\ \forall \boldsymbol{x} \in (B(\boldsymbol{x}_s^*, \delta_2) \setminus \boldsymbol{x}_s^*);\ f(\boldsymbol{x}_s^*) < f(\boldsymbol{x}) \,\}, \tag{14}$$

$$\boldsymbol{X}_i^{*} = \{\, \boldsymbol{x}_i^{*} \mid \exists \boldsymbol{x}_i^{*} \in S,\ \exists \delta_3 > 0,\ \forall \boldsymbol{x}^* \in (X^* \setminus \boldsymbol{x}_i^*);\ \boldsymbol{x}^* \notin B(\boldsymbol{x}_i^*, \delta_3) \,\}. \tag{15}$$

From the above definition, inclusion relations among these sets are easily derived as follows.

$$\boldsymbol{X}^{**} \subset \boldsymbol{X}^*, \quad \boldsymbol{X}_i^* \subset \boldsymbol{X}_s^* \subset \boldsymbol{X}^*, \quad \boldsymbol{X}_i^* \subset \boldsymbol{X}^*. \tag{16}$$

Next, we consider the set such that each element of this set is local minima and is not strict local minima.

Property 10 *The set* $\boldsymbol{X}^* \setminus \boldsymbol{X}_s^*$ *such that its element is local minima and not strictly local minima is formulated as follows.*

$$\boldsymbol{X}_{\bar{s}}^* \equiv \boldsymbol{X}^* \setminus \boldsymbol{X}_s^* = \{\, \boldsymbol{x}_{\bar{s}}^* \mid \exists \boldsymbol{x}_{\bar{s}}^* \in \boldsymbol{X}^*,\ \forall \bar{\delta}_2 > 0,\ \exists \boldsymbol{x} \in (B(\boldsymbol{x}_{\bar{s}}^*, \bar{\delta}_2) \setminus \boldsymbol{x}_{\bar{s}}^*);\ f(\boldsymbol{x}_{\bar{s}}^*) = f(\boldsymbol{x}) \}. \tag{17}$$

From the above fact, it is found the set whose elements are local minima and not strict local minima is function values are all equal at any point on a neighbor of a point of the set. it that there exists a *flat region*. That there is a *flat region*, that incountably infinite local minima exists in this region. On the contrary, we show the following properties hold in case where a flat region not exists.

Property 11 *Necessary and sufficient condition that there is no flat region in* S *(*$\boldsymbol{X}_{\bar{s}}^* = \emptyset$*) is all local minima is strictly local minima (*$\boldsymbol{X}^* = \boldsymbol{X}_s^*$*). Moreover, all local minima is strictly local minima.*

Next, we consider a problem with a flat region.

Example 12 We convert the following problem:

$$(\mathrm{P}^{\underline{F}}) \left\{ \begin{array}{ll} \text{min.} & \tilde{f}(x_1, x_2) = (x_1 - x_2)^2 \\ \text{s.t.} & S = \{(x_1, x_2) \mid (x_1, x_2) \in [0, 1]^2\} \end{array} \right. \tag{18}$$

into the unconstrained minimization problem on $\mathbb{R}^2$ with extended real valued function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by the equation(1). In this problem, f has a flat region on the line segment $x_1 = x_2$ in $(x_1, x_2) \in [0, 1]^2$. Moreover, a point in the flat region is local minimum is local minimum, but is not strictly local minimum, and there is infinite number of local minima on $x_1 = x_2$.

We can shown a relationship between the set of strict local minima $\boldsymbol{X}_s^*$ and the set of isolated local minima $\boldsymbol{X}_i^*$.

Property 13 *An inclusion relation between strict local minima and isolated local minima* ： *All of isolated local minima is strict local minima, that are, the following inclusion relation holds.*

$$\boldsymbol{X}_i^* \subset \boldsymbol{X}_s^*. \tag{19}$$

From inclusion relations (16) and (19), the following relations hold.

Property 14 *In a lower continuous extended real valued functionf* $: \mathbb{R}^n \to (-\infty, \infty]$ *defined by* $S \subset \mathbb{R}^n$*, the following inclusion relations between the sets of minima*$\boldsymbol{X}^{**}$*, local minima*$\boldsymbol{X}^*$ *strictly local minima* $\boldsymbol{X}_s^*$ *and isolated local minima*$\boldsymbol{X}_i^*$*.*

$$\boldsymbol{X}^{**} \subset \boldsymbol{X}^* \subset S, \tag{20}$$

$$\boldsymbol{X}_i^* \subset \boldsymbol{X}_s^* \subset \boldsymbol{X}^* \subset S. \tag{21}$$

From these two relations (20) and (21), we can show the following properties.

Property 15 *If a function* f *has unique local minimum on* $S \subset \mathbb{R}^n$*, the point is strict local minimum and isolated local*

Table 1: A comparison among maximum(max.) number of elements in each types of solution on D^n

types of sets of local minima etc.	notation	max. number of elements
local minima	$\boldsymbol{X}^*$	incountably infinite
local minima not strict local minima	$\boldsymbol{X}^* \setminus \boldsymbol{X}^*_s$	incountably infinite (flat region)
strict local minima	$\boldsymbol{X}^*_s$	countably infinite
isolate loca minima	$\boldsymbol{X}^*_i$	finite
inclusion among sets	$\boldsymbol{X}^*_i \subset \boldsymbol{X}^*_s \subset \boldsymbol{X}^*$	

minimum and f has unique local minimum.

3.2. Finiteness of local minima on a bound set

In this section, we show a property between finiteness of local minima and isolated local minima on bound interval $[a, b]$.

Theorem 16 *A necessary and sufficient condition that the number of local minima is finite on a bound interval $[a, b] \subset \mathbb{R}$ is all local minima on $[a, b]$ are isolated.*

From this theorem and the fact $[a, b] \subset \mathbb{R}$, the following corollary holds.

Corollary 17 *There exists finite local minima on $\mathbb{R}$, then all local minima are isolated.*

We show a property on finiteness of local minima in a hyper rectangle region similar to theorem 16, as follows.

Theorem 18 *A necessary and sufficient condition that the number of local minima is finite on a hyper rectangle region $D^n \equiv \prod_{i=1}^{n}[a_i, b_i] \subset \mathbb{R}^n$ is all local minima on D^n are isolated.*

From this theorem and the fact $D^n \subset \mathbb{R}^n$, the following corollary holds.

Corollary 19 *If there exists finite local minima on $\mathbb{R}^n$, then all local minima are isolated.*

The number of maximum elements in sets of solutions(local minima etc.) are shown in Table 1.

4. Definitions and basic properties of minima and local minima by (connected) level sets

In Sect. 3, we define solutions(local minima etc. and minima) by comparison between function value at a solution and function values at points on neighbor around the solution. In this section, we show definitions of solutions by level set etc. or connected level set.

A notion for "Connectivity" in optimization problem was already defined by Ortega[5, pp.98–100] in 1970s, here are also discussed relations with the uniqueness of the minimum. A definition of connected component that include a point is shown by Dixon et al.[3, p.36].

4.1. Definition of minimum by level set

The following property hold in definition of minimum and its level,

Property 20 *For extended real valued function $f : \mathbb{R}^n \to (-\infty, \infty]$, necessary and sufficient condition that a point $\boldsymbol{x}^{**}$ in minimum.*

$$\forall \varepsilon > 0; \quad L^{\leq}(f(\boldsymbol{x}^{**}) - \varepsilon) = \emptyset. \tag{22}$$

In addition, by using a minimum and its equal level set, the following corollary that is simpler and equivalent to the above property holds,

Corollary 21 *For extended real valued function $f : \mathbb{R}^n \to (-\infty, \infty]$, a necessary and a sufficient condition such that $\boldsymbol{x}^{**}$ is minimum, is that the following equation holds.*

$$L^{\leq}(f(\boldsymbol{x}^{**})) = L^{=}(f(\boldsymbol{x}^{**})). \tag{23}$$

Moreover, we can show the following corollary that is simpler and equivalent to the above property using a minimum and its strict level set,

Corollary 22 *For extended real valued function $f : \mathbb{R}^n \to (-\infty, \infty]$, a necessary and a sufficient condition such that $\boldsymbol{x}^{**}$ is minimum, is that the following equation holds.*

$$L^{<}(f(\boldsymbol{x}^{**})) = \emptyset. \tag{24}$$

4.2. Definitions and basic properties of local minima etc. by (connected) level set

At first, we show an example there exist a region whose local minima is also local maxima.

Example 23 *An problem with the following objective function $\tilde{f}_1 : \mathbb{R} \to \mathbb{R}$*

$$(\mathrm{P}_1)\begin{cases} \text{min.} & \tilde{f}_1(x) = \max\{0, x^2 - 1\}; \\ \text{s.t.} & S = \{x \in \mathbb{R} \,|\, x \in [-3, 3]\}. \end{cases} \tag{25}$$

In this example, there is incountable infinite local minima on $[-1, 1]$, and any point on $(0, 1)$ is also local maxima. In such a problem, local minima and local maxima become to duplicate on its flat region.

We introduce a definition of solutions called "local minimal values set (l.m.v.s.)"[4] for resolving such an unnatural situation.

Definition 24 *For any $\varepsilon > 0$, if there exists $\boldsymbol{x}^* \in \mathbb{R}^n$ and the following equation:*

$$L^{\leq}(f(\boldsymbol{x}^*) - \varepsilon) \cap L^{\leq}_c(f(\boldsymbol{x}^*)) = \emptyset \tag{26}$$

is satisfied, then we call the set $L^{\leq}_c(f(\boldsymbol{x}^*))$ local minimum values set *including $\boldsymbol{x}^*$. And, especially we denote the set $E^*_c(f(\boldsymbol{x}^*))$.*

Definition 25 *All set that is satisfied the Eq.(26) is called* all local minimal values set *$\boldsymbol{E}^*$. That is, all local minimum* *values set* $\boldsymbol{E}^*$ *is formulated as follows.*

$$\boldsymbol{E}^* = \{\boldsymbol{x}^* \mid \forall\varepsilon > 0;\ L^{\leq}(f(\boldsymbol{x}^*) - \varepsilon) \cap L_c^{\leq}(f(\boldsymbol{x}^*)) = \emptyset\}. \quad (27)$$

Inclusion relations among $\boldsymbol{E}^*$ and set of the previous solutions are as follows.

$$\boldsymbol{X}_s^* \subset \boldsymbol{E}^* \subset \boldsymbol{X}^*. \quad (28)$$

In addition, we show equivalent two definitions to l.m.v.s. $L_c^{\leq}(f(\boldsymbol{x}^*))$.

Property 26 *Necessary and sufficient condition that* $L_c^{\leq}(f(\boldsymbol{x}^*))$ *is l.m.v.s. is the following equation holds.*

$$\forall \boldsymbol{y} \in L_c^{\leq}(f(\boldsymbol{x}^*)); \quad f(\boldsymbol{y}) = f(\boldsymbol{x}^*). \quad (29)$$

From Eq. (8), (10), (11) and the above property, the following corollary also hold.

Corollary 27 *Necessary and sufficient condition that* $L_c^{\leq}(f(\boldsymbol{x}^*))$ *is l.m.v.s. is one of the following two holds.*

$$L_c^{\leq}(f(\boldsymbol{x}^*)) = L_c^{=}(f(\boldsymbol{x}^*)), \quad (30)$$

$$L_c^{<}(f(\boldsymbol{x}^*)) = \emptyset. \quad (31)$$

4.3. Definitions of multimodal and weak unimodal functions

Definition 28 *Let the* number of connected components *for all l.m.v.s. that is determined for a function* $f : \mathbb{R}^n \to (-\infty, +\infty]$ *is called by a* (lower) number of modal*, and is denoted by* $\#|\boldsymbol{E}^*|_c$.

Definition 29 *From the definition,* weak (lower) unimodal functions *and* (lower) multimodal functions *are defined as follows.*

$$\begin{cases} \#|\boldsymbol{E}^*|_c = 1, & \text{weak lower unimodal function;} \\ \#|\boldsymbol{E}^*|_c \geq 2, & \text{lower multimodal function.} \end{cases} \quad (32)$$

In the problem P_1 of Eq.26, f_1 is a convex function and l.m.v.s. is $\boldsymbol{E}^* = \{[-1, 1]\}$. Thus, $\#|\boldsymbol{E}^*|_c = 1$ and f_1 become to a weakly unimodal function on $[-1, 1]$.

All l.m.v.s. of the next problem shown in Fig.1 is $\boldsymbol{E}^* = \{1, 4, [10,11]\}$, and the number of modal $\#|\boldsymbol{E}^*|_c = 3$. Thus this function is multimodal function on [0.12].

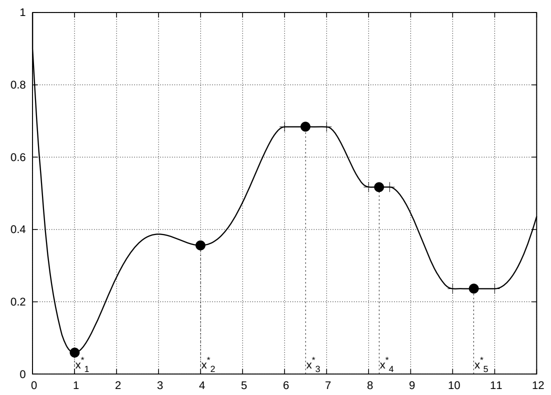

Fig.1 A function f with three flat regions

Solution sets(set local minima(min.) etc.) and each solution set for the problem of Fig. 1 is shown in Table 1. In

Table 2: Solution sets(set local minima(min.) etc.) and each solution set for the problem in Fig. 1

types of solutions	set in Fig.1
min.	$\boldsymbol{X}^{**} = \{1\}$
local min.	$\boldsymbol{X}^* = \{1, 4, (6,7), [8,8.5), [10,11]\}$
isolates loca min.	$\boldsymbol{X}_i^* = \{1, 4\}$
l.m.v.s.	$\boldsymbol{E}^* = \{1, 4, [10,11]\}$

this table, the number of connected components for the set of local minima $\boldsymbol{X}^*$ is $\#|\boldsymbol{X}^*|_c = 5$. But point x_3^* is not only a local minimum but also a local maximum, and x_3^* is also a stationary point. The above duplicated situation, obviously unnatural.

5. Conclusions

In addition three types of local solutions, it is clear that the set with flat region is obtained by excepting strict local minima from local minima. Moreover, if there exists a finite local minima on a interval or a hyper-rectangle region then all local minima is isolated, and its converse proposition also hold.

We introduce the definition of local minimum values set (l.m.v.s.) by connected level set besides well known three types of local minima. In addition, it is clear that three other equivalent definitions to l.m.v.s. are exist. In generally, definitions by level set be able to represent more simply than the traditional definitions using neighbor of a solution.

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