Two coexisting two-dimensional tori generated in a three-coupled delayed logistic map

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Abstract: Quasi-periodic bifurcations have attracted considerable attention in recent years. In this study, we discuss two coexisting two-dimensional tori in an Arnol'd tongue generated in a three coupled delayed logistic map. The two coexisting two-dimensional tori comprise 93 invariant closed curves. One of two-dimensional tori disappear by a quasi-periodic saddle-node bifurcation, and the other two-dimensional torus bifurcates to a three-dimensional torus via a quasi-periodic saddle-node cycle bifurcation. The generation of the three-dimensional torus is confirmed by observing the attractor on a double Poincaré section.

Keywords—**Quasi-periodic oscillations, Quasi-periodic bifurcatioins Latex, Coupled delayed logistic map**

1. Introduction

Quasi-periodic oscillations and quasi-periodic bifurcations been the subjects of intensive research in recent years [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31]. Vitoro *et al.* demonstrated that there are two possible bifurcation routes from a two-torus to a three-torus. One is a quasi-periodic Hopf bifurcation, and the other is a quasi-periodic saddle-node bifurcation [10]. Quasi-periodic Hopf bifurcations are also called quasi-periodic Neimark–Sacker (NS) bifurcation.

One of the major concerns in studying Arnol'd tongues in dynamics that can generate three- or higher quasi-periodic attractors is to find out how an Arnol'd tongue transits to a higher-dimensional Arnol'd tongue near quasi-periodic Hopf bifurcation. Takens and Wagener conducted a bifurcation analysis for such complex bifurcation [27] Kuznetsov and Meijer conducted Lyapunov analysis and clarified the bifurcation structure near a codimension-two bifurcation point they named flip-NS bifruction [28]. Broer *et al.* reported more complex transitions [1].

In this study, we investigate complex quasi-periodic bifurcations for two coexisting two-dimensional tori in an Arnol'd tongue generated by a three-coupled delayed logistic map. We find a novel bifurcation structure where there exists an conventional Arnold'd tongue wherein two periodic attractors with period 93 coexist. This Arnol'd tongue bifurcates to a two-dimensional torus-resonance tongues in which two two-dimensional tori that comprise 93 invariant closed circles (ICCs) coexist.

One of these ICCs disappear via a quasi-periodic saddle-node bifurcation, and the other bifurcates to a three-dimensional torus quasi-periodic due to a quasi-periodic saddle-node cycle bifurcation. We confirmed these complex quasi-periodic bifurcations by illustrating one-parameter bifurcation diagrams.

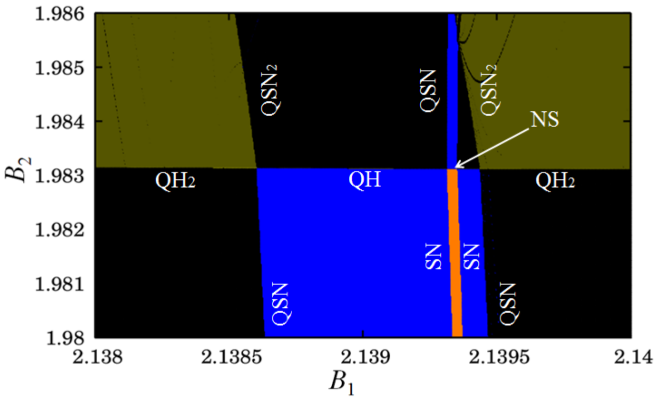

Figure 1. Two-parameter Lyapunov diagram near the QH_2 bifurcation curve ($M = 10,000,000$ and $N = 10,000,000$ with a grid mesh of $1,000 \times 1,000$).

2. Analysis

We carry out a Lyapunov analysis of the three-coupled delayed logistic map expressed by the following equation.

$$
\begin{aligned}
&F((x_n, y_n, z_n, w_n, u_n, v_n)^\top) : \\
&x_{n+1} = y_n, \\
&y_{n+1} = B_1 y_n(1 - x_n) + \varepsilon_1 w_n + \varepsilon_2 v_n, \\
&z_{n+1} = w_n, \\
&w_{n+1} = B_2 w_n(1 - z_n) + \varepsilon_3 v_n + \varepsilon_4 y_n, \\
&u_{n+1} = v_n, \\
&v_{n+1} = B_3 v_n(1 - u_n) + \varepsilon_5 y_n + \varepsilon_6 w_n,
\end{aligned}
\tag{1}
$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$, and ε_6 are coupling parameters. Because the single delayed logistic map exhibits an invariant one-torus that corresponds to a two-dimensional torus in vector fields via an NS bifurcation, the three-coupled delayed logistic map can generate an invariant three-torus that corresponds to a four-dimensional torus in vector fields with three zero Lyapunov exponents. Throughout the this study, we fix

$$
\begin{aligned}
&\varepsilon_1 = 0.01, \ \varepsilon_2 = 0.002, \ \varepsilon_3 = 0.001, \ \varepsilon_4 = 0.02, \\
&\varepsilon_5 = 0.01, \ \varepsilon_6 = 0.01, \ B_3 = 2.05,
\end{aligned}
\tag{2}
$$

and allow parameters B_1 and B_2 to vary.

The six Lyapunov exponents in Eq. (1) are calculated by the following procedure.

$$
\begin{aligned}
&\lambda_1 \simeq \frac{1}{N}\sum_{j=M+1}^{M+N} \ln ||DF_j e_1^j|| \\
&\lambda_1+\lambda_2 \simeq \frac{1}{N}\sum_{j=M+1}^{M+N} \ln ||DF_j e_1^j \times DF_j e_2^j|| \\
&\lambda_1+\lambda_2+\lambda_3 \simeq \\
&\frac{1}{N}\sum_{j=M+1}^{M+N} \ln ||DF_j e_1^j \times DF_j e_2^j \times DF_j e_3^j|| \\
&\lambda_1+\lambda_2+\lambda_3+\lambda_4 \simeq \\
&\frac{1}{N}\sum_{j=M+1}^{M+N} \ln ||DF_j e_1^j \times DF_j e_2^j \times DF_j e_3^j \times DF_j e_4^j|| \\
&\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5 \simeq \\
&\quad \frac{1}{N}\sum_{j=M+1}^{M+N} \ln ||DF_j e_1^j \times DF_j e_2^j \times DF_j e_3^j \\
&\times DF_j e_4^j \times DF_j e_5^j|| \\
&\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6 \simeq \frac{1}{N}\sum_{j=M+1}^{M+N} \\
&\ln ||DF_j e_1^j \times DF_j e_2^j \times DF_j e_3^j \\
&\times DF_j e_4^j \times DF_j e_5^j \times DF_j e_6^j||
\end{aligned}
\tag{3}
$$

where M and N are sufficiently large integers.

In the figure, orange, blue, black, and dark green denote regions generating a periodic solution, an invariant one-torus, an invariant two-torus, and an invariant three-torus, respectively. In addition, NS, QH, SN, QSN indicate a Neimark–Sacker bifurcation, a quasi-periodic Hopf bifurcation, a saddle-node bifurcation, and a quasi-periodic saddle-node bifurcation curve, respectively. A magnified view of Fig. 1 is shown in Fig. 2. At a point marked P, two periodic solutions with period 93 are observed. Moreover, two invariant one-tori that comprise 93 ICCs are observed at a point Q. Figure 3 shows a one-parameter bifurcation diagram in which

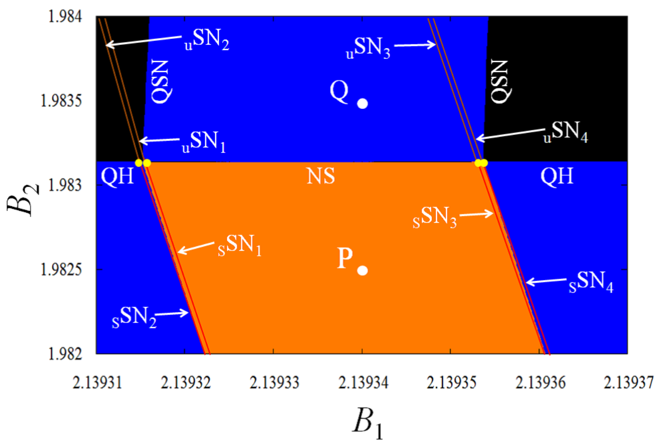

Figure 2. Magnified view of Fig. 1 .

one of the attractors is colored in red, and the other is colored in green, which is traced to left from the point P. One of the

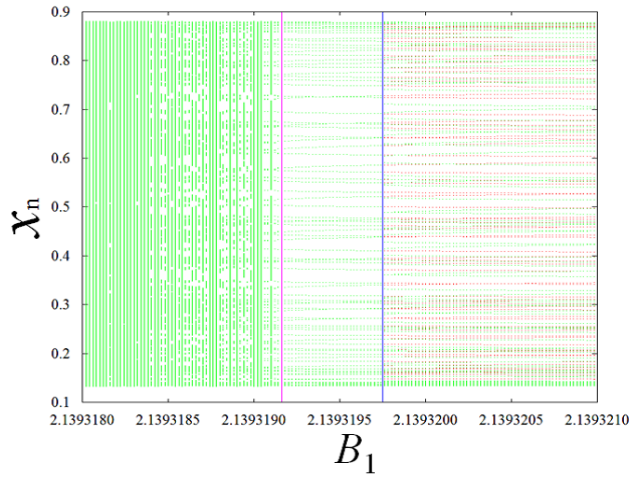

Figure 3. One-parameter bifurcation diagram. The bifurcation parameter B_1 decreases from P.

periodic solutions with period 93 denoted in red disappears owing to an SN bifurcation at a point marked blue line, which is derived rigorously by the procedure presented in [36]. In contrast, the other periodic solution denoted by green bifurcates to an invariant one-torus that comprises 93 ICCs at the red line, which can be an SN cycle bifurcation. The attractor in the state space is shown in Fig. 4. The solution is identified as an invariant one-torus because the dynamics have one exact zero Lyapunov exponent. In addition, we trace the bi-

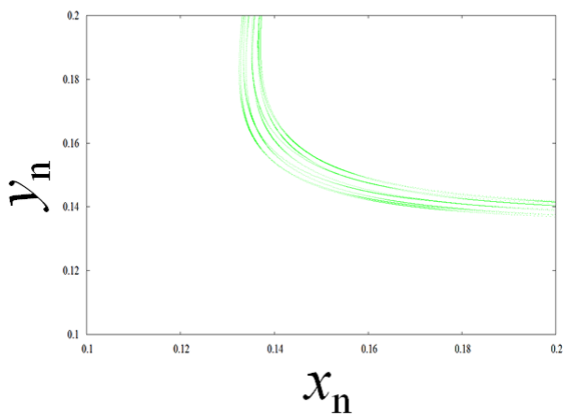

Figure 4. Invariant one-torus that is observed after the saddle-node cycle bifurcation.

furcation parameter from P to right. In this case, the periodic solution denoted in green disappears by an SN bifurcation denoted in the red line as shown in Fig. 5, and furthermore, the other periodic solution bifurcates to an invariant one-torus via an SN cycle bifurcation denoted by the blue line.

In contrast, by decreasing the parameter B_1 from Q, one-parameter bifurcation diagram shown in Fig. 6 is obtained. One of the invariant one-tori that comprises 93 ICCs denoted in red disappears owing to a QSN bifurcation. Furthermore,

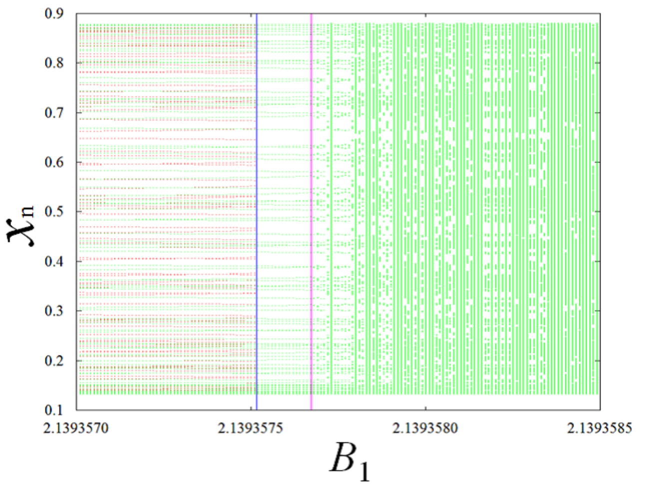

Figure 5. One-parameter bifurcation diagram. The bifurcation parameter B_1 increases from P.

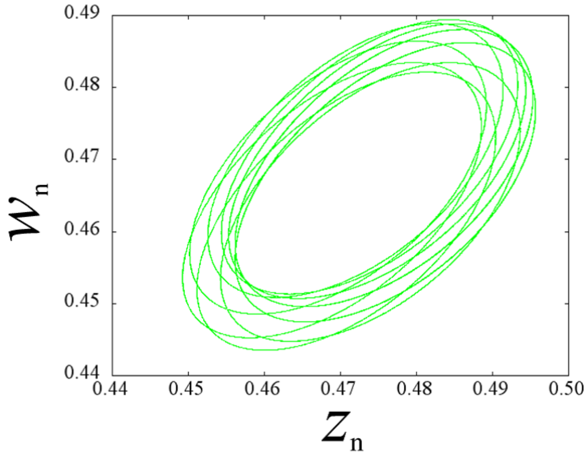

Figure 7. Invariant two-torus on the double Poincaré section that is observed after the quasi-periodic saddle-node cycle bifurcation.

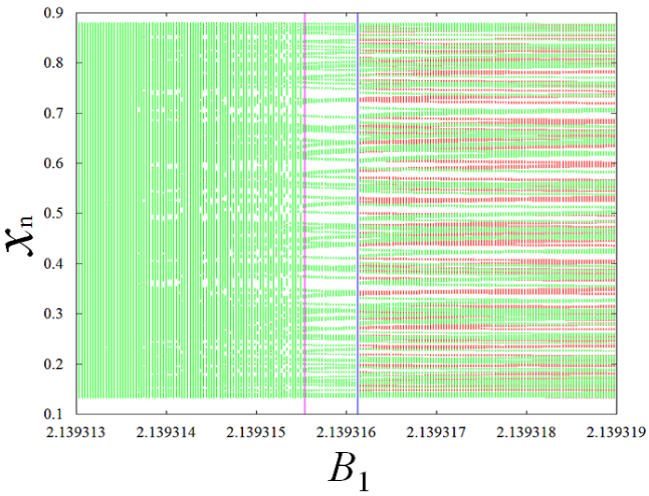

Figure 6. One-parameter bifurcation diagram. The bifurcation parameter B_1 decreases from Q.

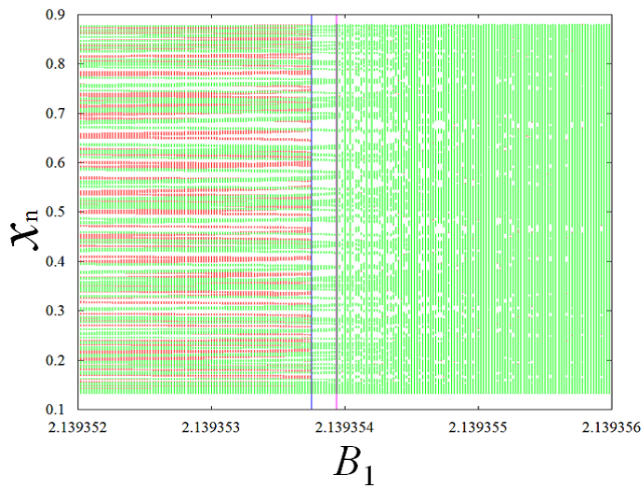

Figure 8. One-parameter bifurcation diagram. The bifurcation parameter B_1 increases from Q.

the other invariant one-torus denoted in green bifurcates to an invariant two-torus via QSN cycle bifurcation. The attractor after the quasi-periodic saddle-node bifurcation is identified as an invariant two-torus because the attractor on the double Poincaré section forms an ICC as shown in Fig. 7. The bifurcation parameter values at which the QSN bifurcation and the QSN cycle bifurcation can be obtained by observing the attractors.

Finally, one-parameter bifurcation diagram that is obtained by tracing the bifurcation parameter B_1 to right from Q is presented in Fig. 8. In this case, an invariant one-torus denoted in red disappears at the blue line in the figure via a quasi-periodic saddle-node bifurcation. Furthermore, the other invariant one-torus denoted in green bifurcates to an invariant two-torus through a quasi-periodic saddle-node cycle bifurcation.

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