Conversion of stability observed in Van der Pol oscillator by unstable time-delayed controller

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Abstract—We have demonstrated that an unstable time-delayed controller can provide a tool to control the stability of Hopf bifurcation. The unstable time-delayed controller has used to stabilize an unstable periodic orbit operating in subcritical mode. By developing a new coupling method the unstable time-delayed system was improved to be applied for conversion of stability in Hopf bifurcation system. In this study, as an example, conversion of stability of Van der Pol oscillator has been confirmed both experimentally and numerically. The control method would control various oscillations of systems arising from Hopf bifurcation.

1. Introduction

Time-delayed systems in nonlinear system takes place much complicated phenomena because of their infinite degree of freedom. Many engineers and physicists have tackled to elucidate physically fundamental properties and to develop applications using its various phenomena. The most brilliant innovation was the time-delayed feedback control method [1] developed in the early nineties. The method makes a chaotic attractor converge to an unstable periodic orbit only by imposing a time-delayed signal to a chaotic system to be controlled. The great advantage points are its robustness and ease of handle. However, the method also has an intrinsic limitation that it cannot be applied for torsion free systems [2]. To overcome such limitation the unstable time-delayed control method was developed [3]. By intuitive understanding, a redundant torsion is introduced artificially by attaching an unstable controller. The trial was succeeded in control of torsion free systems but the controllability is not enough for applications because of its narrow operating margin. From the engineering viewpoint we have developed to expand operating margin so as to stand for practical applications. We choose sigmoidal function, $\mathrm{sgn}(x)$, as feedback function instead of linear function, x. The modified unstable time-delayed feedback method derived extremely wide operating margin and then accompanied conversion of the stability as something like a side effect. In this manuscript we will describe experimental and numerical estimations of the conversion as a phenomenology.

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2. Van der Pol oscillator with unstable time-delayed controller

We consider a delay-coupled Van der Pol oscillator with coupling constnt K,

$$\begin{aligned}\ddot{x} \pm \left(\epsilon_0 - x^2\right)\dot{x} + \omega_0^2 x &= K\dot{u}, &(1)\\ \ddot{u} - \left(\epsilon_c - u^2\right)\dot{u} + \omega_c^2 x &= -K\left(\dot{x} - \dot{x}_\tau\right), &(2)\end{aligned}$$

where $\dot{x}_\tau = \dot{x}(t-\tau)$ and the upper and lower sign in eq. (1) denote the sub- and supercritical Hopf bifurcation, respectively. Near the supercritical bifurcation point, the system changes from the fixed point to the stable periodic orbit as the bifurcation parameter ϵ_0 changes from negative to positive. In the case of the subcritical Hopf bifurcation, the unstable fixed point changes to the stable fixed point with the bifurcation parameter. In polar coordinate, if variables are transformed as $x = r\cos\phi, \dot{x} = -\omega_0 r\sin\phi, u = w\cos\psi$ and $\dot{u} = -\omega_c w \sin\psi$, eqs. (1) and (2) can be written by

$$\begin{aligned}\dot{r} &= \pm\left(\frac{\epsilon_0}{2} - \frac{r^2}{8}\right)r + \frac{K\omega_c}{2\omega_0} w\cos(\phi-\psi)\\ \dot{\phi} &= \omega_0 - \frac{K\omega_c}{2\omega_0}\cdot\frac{w}{r}\sin(\phi-\psi)\\ \dot{w} &= \left(\frac{\epsilon_c}{2} - \frac{r^2}{8}\right)w - \frac{K\omega_0}{2\omega_c} r\cos(\phi-\psi)\\ &\quad + \frac{K\omega_0}{2\omega_c} r_\tau \cos(\phi_\tau - \psi)\\ \dot{\psi} &= \omega_c + \frac{K\omega_0}{2\omega_c}\cdot\frac{r}{w}\sin(\phi-\psi)\end{aligned}$$

$$+\frac{K\omega_0}{2\omega_c}\cdot\frac{r_\tau}{w}\sin(\phi_\tau-\psi),$$

where we take average for variation of phases. Near the Hopf bifurcation point, we fix ϵ_c as a small positive value and we assume that variation of $r(t)$ and $w(t)$ are very slow in comparison with the phase $\phi(t)$ and $\psi(t)$. Moreover, in the case in which the difference between ω_0 and ω_c is nearly equal to zero, the time derivative of the phase ϕ is constant because the difference between phases, $\theta=\phi-\psi$, can be a slowly varying quantity. Namely, the delayed phase can be written as

$$\begin{aligned}\phi_\tau &\sim \phi-\tau\dot{\phi}\\ &\sim \phi-\left\{\omega_0-\frac{K\omega_0}{2\omega_c}\cdot\frac{w}{r}\sin\theta\right\}.\end{aligned}$$

Taking into account the situation considered above, we obtain the equations for the amplitudes and the phase difference as follows:

$$\begin{aligned}\dot{r} &= f_\pm(r)+\frac{K\omega_c}{2\omega_0}w\cos\theta\\ \dot{w} &= \frac{\epsilon_c}{2}w-\frac{K\omega_0}{2\omega_c}r\cos\theta\\ &\quad +\frac{K\omega_0}{2\omega_c}r_\tau\cos\left(\theta+\frac{K\omega_c\tau}{2\omega_0}\cdot\frac{w}{r}\sin\theta\right)\\ \dot{\theta} &= \Delta\omega+\frac{K}{2}\left(\frac{\omega_0 r}{\omega_c w}-\frac{\omega_c w}{\omega_0 r}\right)\sin\theta\\ &\quad -\frac{K\omega_0 r_\tau}{2\omega_c w}\sin\left(\theta+\frac{K\omega_c\tau}{2\omega_0}\cdot\frac{w}{r}\sin\theta\right),\end{aligned}$$

where

$$\begin{aligned}f_\pm(r) &= \pm\left(\frac{\epsilon_0}{2}-\frac{r^2}{8}\right)r^2\\ \theta &= \phi-\psi\\ \Delta\omega &= \omega_0-\omega_c\end{aligned}$$

If $\Delta\omega\sim 0$, θ varies slowly and the phase locking occurs at $\theta^*=0$ and $\theta^*=\pi$. After phase locking, amplitudes are slowly attracted to the steady states,

$$(w^*,r^*)=\begin{cases}(0,0)\\ \left(0,\sqrt{\frac{\epsilon_0}{2}}\right)\end{cases}$$

with linear evolution, $r_\tau\sim r-\tau\dot{r}$. In this case, the amplitude equation can be approximated as

$$\dot{r} = f_\pm(r)+w' \tag{3}$$

$$\dot{w}' = \frac{\epsilon_c}{2}w'-\kappa\left(r-r_\tau\right), \tag{4}$$

where $w'=(K/2)\cos\theta^* w$ and $\kappa=(K^2/4)\cos^2\theta^*$. Equations (3) and (4) can be written as

$$\ddot{r}+\left\{\kappa\tau-f'_\pm(r)-\frac{\epsilon_c}{2}\right\}\dot{r}+\frac{\epsilon_c}{2}f_\pm(r)=0.$$

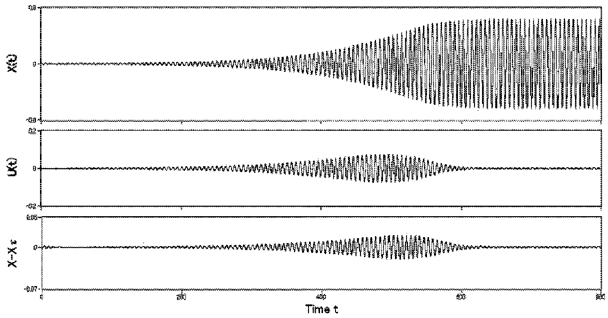

Figure 1: Results of LabView simulation for a sub-to-super conversion. The target $x(t)$, the control $u(t)$ and the difference between $x(t)$ and $x(t-\tau)$ are shown. The control is switched on at $t=20$.

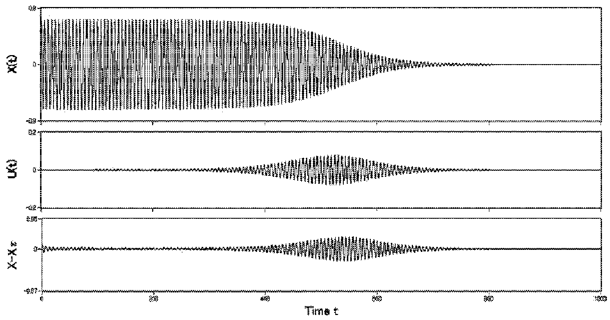

Figure 2: Results of LabView simulation for a super-to-sub conversion. The target $x(t)$, the control $u(t)$ and the difference between $x(t)$ and $x(t-\tau)$ are shown. The control is switched on at $t=100$.

Because of $\ddot{r}\sim 0$, $\epsilon_c>0$ and $\kappa\tau>f'_\pm(r)+\epsilon_c/2$, we obtain

$$\dot{r}\sim -f_\pm(r)=f_\mp(r). \tag{5}$$

This means that super-/sub- critical system can be converted to sub-/super- critical system.

We carried out the simulation by use of LabView. The parameters are set as $(\epsilon_0,\epsilon_c,\omega_0,\omega_c,\tau)=(0.1,0.1,1.0,1.0,2\pi)$. Results are shown in Figs. 1 and 2. In Fig. 1, the subcritical system is converted to the supercritical system by the control switched on at $t=20$ with $K=0.4$ and then we can obtain the stable limit cycle from the fixed point. In Fig. 2, the supercritical system is converted to the subcritical system by the control switched on at $t=100$ with $K=0.4$ and then we can obtain the fixed point from a stable limit cycle. This is regarded as an amplitude deth phenomenon occurs. Thus, we can comfirm the varidity of the theory menshioned above by numerical simulations.

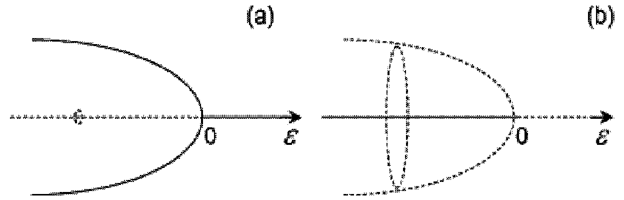

Figure 3: Bifurcation diagrams of Van der Pol oscillators operating in (a) supercritical mode and (b) sub critical mode. In both cases unstable states for $\epsilon < 0$, illustrated by dashed line, can be stabilized by the time-delayed unstable controller.

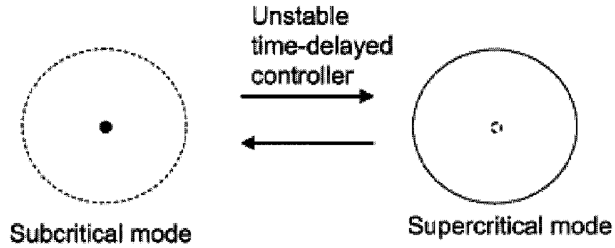

Figure 5: The conceptual figure of the conversion of the stability in Hopf bifurcations. Both modes can be changed into opposite modes with the unstable time-delayed controller.

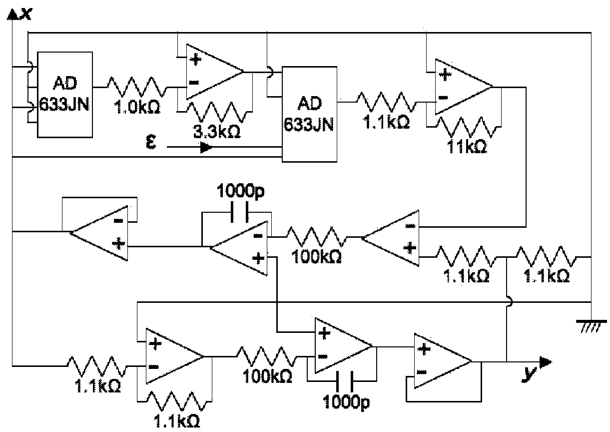

Figure 4: The circuit diagram representing Van der Pol oscillator. The circuit consists of OP-amps and multipliers.

3. Experimental system

The conversion of stability in Hopf bifurcation system should be realized in experiment. Let us start from the following Van der Pol system :

$$\dot{x} = -y \pm \left(\epsilon x + \frac{x^3}{3}\right) \quad (6)$$

$$\dot{y} = x, \quad (7)$$

Here ϵ represents a bifurcation parameter of Van der Pol system. The values of sign in eq. (6) corresponds to the operations in supercritical and subcritical modes, as illustrated in Fig. 3. In supercritical mode, the system has an unstable steady state at the origin for $\epsilon < 0$. On the other hands, an unstable orbit is hidden in subcritical mode for $\epsilon < 0$. A circuit diagram representing Van der Pol equation is shown in Fig. 4.

The both unstable states can be stabilized by the following algorism with a time-delayed unstable controller.

$$\dot{x} = -y(t) \pm \left(\epsilon x + \frac{x^3}{3}\right) + w \cdot f(x) \quad (8)$$

$$\dot{y} = x \quad (9)$$

$$\dot{w} = \lambda_c w - K\left(x - x_\tau\right) f(x), \quad (10)$$

where λ_c and K denote the strength of instability of the unstable controller and coupling strength, respectively. $f(\cdot)$ is feedback function which is selected as $\mathrm{sgn}(x)$ mentioned above. The delay time τ should be chosen to be fundamental period of the target system.

Here it should be remarked that the state of Van der Pol oscillator is never invaded by the control signal $w \cdot f(x)$ after the control is achieved since control signal w converges to zero. In our investigation the conversion of operating modes shown in Fig. 5 was confirmed.

4. Results of experiments

In experiments the system written in eqs. (8)-(10) was realized with many liner integrated circuits such as operational amplifiers except for a delay element constructed mainly from a FIFO system. The set of parameters was chosen to be $(\epsilon, \lambda_c, K, \tau) = (-0.1, 0.05, 0.1, 2\pi)$. We would like to show here the case of supercritical operation mode (see ref [4] for the subcritical operation mode), where periodic orbit would converge to an unstable steady state located at the origin. Figure 6 shows the time series of $x(t)$ and $w(t)$, where T means a normalized time which differ from measured time by time constant of integrators implemented. Here the time series can be separated into three phases as follows: Phase I, $T < 0$, the state values were fixed at the initial condition $(x, y, w) = (0, 0, 0)$; Phase II, $50 \geq T > 0$, the system operated in the free running mode whose behavior obeys eqs. (6) and (7) ; Phase III, $T \geq 50$, the unstable control was applied to the system. As shown in Fig. 6, while the oscillation grew from the origin in Phase II because supercritical mode has the stable periodic orbit, the oscillation disappeared in Phase

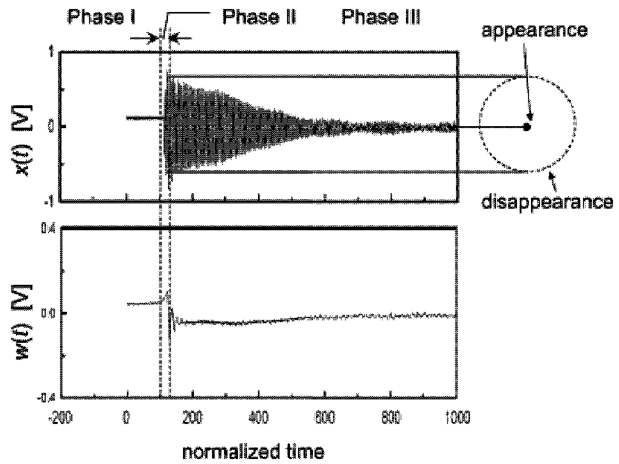

Figure 6: The control process of van der Pol oscillator operating in supercritical mode with the time-delayed feedback controller. After onset of the control, i.e. in Phase III, the stable oscillation disappeared although w vanished.

III although control signal w converges to zero. This means that the unstable controller changed from the unstable steady state located in the origin to the stable one. In other words, supercritical operation was converted equivalently to subcritical operation.

5. Conclusion

From the results employing Van der Pol oscillator as an example, we guess that the time-delayed unstable controller has a possibility to control oscillations in wide field which come from Hopf bifurcation. In order to verify experimental results, numerical simulations by solving eqs. (9)-(10) with 4th order Runge-Kutta method should be executed. Furthermore, we have a plan to study on the possibility of the unstable control for chaotic oscillations such as Lorentz system. These investigations are still in progress.

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