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Study on several graph coloring problems related with wireless networks

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I. Introduction

Currently, spread such as smartphones or products equipped with communication functions, the use of wireless communication has increased. Along with the increase, its use is also wide-ranging. In wireless network, each channel assigns to each communication. Then, if the same channel uses at close range, comfortable communication can not be performed by interference. To consider this problem, it is known widely how to replace the channel assignment problem to the coloring problem of graphs.

Generally, the coloring problem in graph theory, vertices or edges of a graph are assigned colors such that no two adjacent vertices or edges have the same color. Reducing the number of colors in the graph associate reducing the number of channels used in the wireless network.

When we consider a graph coloring as a model of communication, we must choose a suitable coloring condition that communication methods and performance of the terminal are assumed. In usual, the model of wireless network is the strong edge coloring such a way as to prevent interference. The strong edge coloring of the graph is an assignment of colors to the edges such that every two edges of distance at most two receive different colors. To use the strong edge coloring is due to that co-channel interference dose not occur. But co-channel interference may not occur in the conditions of it. Therefore, it has been proposed a number that different colorings being more tightly or considering the performance of the terminal.

In this paper, we pick up some colorings and a extension of them, show the minimum number of colors (called the chromatic number) on the graph whose degrees are constant.

II. Definition

We define some of the terms for discussion. $G = (V(G), E(G))$ is an undirected graph such that $V(G)$ is the set of vertices and $E(G)$ is the set of edges, $|V(G)|$ and $|E(G)|$ is the number of each elements. The degree of a vertex v in a graph, denoted by $deg(v)$, is the number of edges incident with v. Tree T is the graph that is connected and has no cycle. In this paper, we use the tree where the degree of all vertices except the endpoints is constant value d and the size of the tree is enough large. It is called the *d-constant tree*, denoted by T_{d} .

Path $p(u,v)$ is a sequence of edges which connect a sequence of vertices between two vertices u and v, the distance $dist(u,v)$ is the number of edges in the shortest path between u and v. The distance $dist(e,e')$ between two edges $e = (u,v)$ and $e' = (w,q)$ is defined as $\min\{dist(u,w), dist(u,z), dist(v,w), dist(v,z)\}$. Therefore, the distance $dist(v,e')$ between a vertex v and an edges $e' = (w,z)$ is defined as $\min\{dist(v,w), dist(v,z)\}$ (Fig.1). We use the structure of an ℓ-*ball* to consider the chromatic number, the following is the definition of it.[1]

Definition 1. For a graph G and an integer $\ell \geq 0$, we define an ℓ-*ball* as a maximum subgraph $S_\ell \subseteq G$ such that every two edges of S_ℓ are at distance ℓ or less from each other .

When we build S_ℓ , to start at the middle of S_ℓ and move outward. This middle of S_ℓ is denoted by C_ℓ, define as a set of a vertex if ℓ is even, or two vertices joined by an edge if ℓ is odd. Since T_{d} does not have a cycle, an ℓ-ball S_ℓ is the subgraph induced by the set of vertices that are at distance of less than or equal to $\lfloor \frac{\ell}{2} \rfloor + 1$ from C_ℓ (Fig.2). In other words, if $Q = \bigcup_{x \in C_\ell} \{u \in V(G), dist(u,x) \leq \lfloor \frac{\ell}{2} \rfloor + 1\}$, the $\ell - ball$ S_ℓ is $S_\ell = G[Q]$ that is the subgraph induced in G by Q. Let e, e' be edges that incident endpoints of S_ℓ , $dist(e, C_\ell)$ = $dist(e', C_\ell)$ = $\lfloor \frac{\ell}{2} \rfloor$, and if $p(e,e')$ has all C_ℓ, certainly $dist(e,e') = \ell$. That is, farthest two edges in S_ℓ always through C_ℓ. There is no ambiguity, the ℓ-*ball* in the graph will be denoted S_ℓ .

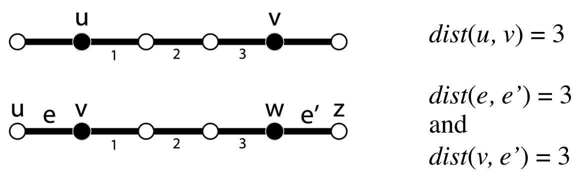
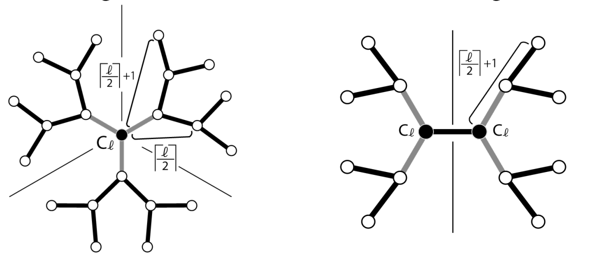


Fig. 1. The distance of vertices and edges.



$\ell = 4$ (even) $\ell = 3$ (odd)

Fig. 2. The ℓ-*ball* and the distance from C_ℓ. ($d = 3$)

III. CHROMATIC NUMBER OF D-CONSTANT TREE

3.1 ℓ -distance edge coloring

For an integer $\ell \geq 0$, the ℓ-distance edge coloring of the graph is an assignment of colors to the edges such that any two edges *e* and *e'* with $dist(e,e') \leq \ell$ have different colors.[1] (Fig.3). A 0-distance edge coloring is the ordinary edge coloring and a 1-distance edge coloring is the strong edge coloring. This coloring has been mainly assumed the sensor network[2]. In the case of the sensor network, it is considered the situation, such as a number of terminals exist in the communication range of the terminal.

The chromatic number is the minimum number of colors required to assign to all edges in the graph. The following theorem gives the chromatic number of the ℓ-distance edge coloring of T_d.

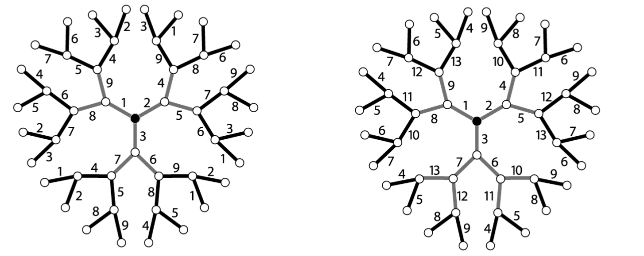
Theorem 1. Let T_d be the *d-constant tree*, $\ell \geq 0$. Then, the ℓ-chromatic number $\chi'_\ell(T_d)$ is given by:

$$\chi'_\ell(T_d) = |E(S_\ell)| = \begin{cases} \sum_{i=0}^{\lfloor \ell/2 \rfloor} d(d-1)^i & (\ell : \text{even}) \\ 2\sum_{i=0}^{\lfloor \ell/2 \rfloor + 1} (d-1)^i - 1 & (\ell : \text{odd}) \end{cases} \quad (1)$$

Proof. Fist of all, we consider the size by ℓ*-ball* S_ℓ. The number of edges from C_ℓ within $\lfloor \frac{\ell}{2} \rfloor + 1$, through one incident edge of C_ℓ is $1+(d-1)+(d-1)^2+...+(d-1)^{\lfloor \frac{\ell}{2} \rfloor}$. C_ℓ's incident edges equal to *d* (ℓ:even) and 2(*d*-1) (ℓ:odd). (The gray edges in Fig.2.) Therefore, $|E(S_\ell)|$ is (1).

From *Def.1*, the edges of S_ℓ should be assigned different colors each other. Then, it shows that we can color the whole in T_d by $|E(S_\ell)|$ colors only. The following will be described for the case of T_d is even.

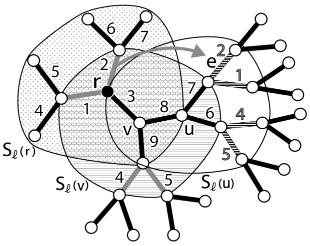
Let a vertex *r* of T_d select as C_ℓ and corresponding ℓ-*ball* that is denoted by $S_\ell(r)$ is assigned colors. Next, we select the adjacent vertex *r'*, and assign colors to $S_\ell(r')$. We repeat this coloring around vertices already chosen as C_ℓ. Herein we can see that two subgraphs $S_\ell(v)$ and $S_\ell(u)$ can share some edges, and $S_\ell(v)$ has already been assigned colors. Let *e* be a uncolored edge of $S_\ell(u)$, then *e* is the farthest edge of $S_\ell(u)$ from edges already colored without $S_\ell(u)$. The distance between *e* and them is over ℓ because the pass through the C_ℓ. And edges within a distance ℓ from *e* is uncolored without belonging to S_ℓ. Therefore, when we assign color to *e*, we can use the rest of a set of $|E(S_\ell)|$ colors completely (Fig.4). Hence, it is possible to color the whole T_d. The case of the odd is also same way but C_ℓ is two vertices, we consider it with respect to the edge of C_ℓ.



$\ell = 2, d = 3$
$\chi'_\ell(T_d) = d + d(d-1) = 9$

$\ell = 3, d = 3$
$\chi'_\ell(T_d) = 2\{1+(d-1)+(d-1)^2\} - 1 = 13$

Fig. 3. The ℓ-distance edge coloring.



Stripe edges of $S_\ell(u)$ can be assigned colors of gray edges of $S_\ell(v)$.

Fig. 4. The way of coloring by ℓ-distance.

3.2 *f*-edge coloring

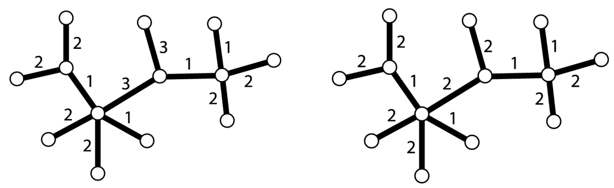
For a integer $f \geq 0$, the *f*-edge coloring of a graph colors edges so that $deg(v,c) \leq f$ where $deg(v,c)$ in a graph is the number of edges incident with *v* and that have color *c*.

In other words, it can be assigned the same color to *f* edges in each vertices (Fig.5). 1-edge coloring is the general edge coloring. This coloring is considered to be applied to the scheduling problem for parallel processing[3]. The following theorem gives $\chi'_\ell(T_d)$.

Theorem 2. Let T_d be the *d-constant tree*, $f \geq 1$. Then, the *f*-chromatic number $\chi'_\ell(T_d)$ is given by:

$$\chi'_f(T_d) = \left\lceil \frac{d}{f} \right\rceil. \quad (2)$$

Proof. If a distance of two vertices is more than or equal to 2, these can be assigned a same color. So we assume the coloring of 0-ball on *f*-edge coloring. Since this coloring can be assigned the same color to *f* edges, certainly 0-ball is able to be coloring by (2). The coloring process is the same as Theorem1. Consequently, it is possible to color the all edges of T_d.



f = 2 $\chi'_f = 3$ f = 3 $\chi'_f = 2$

Fig. 5. The *f*-edge coloring.

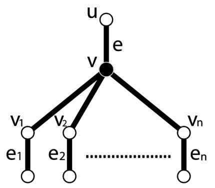
3.3 CIR_ℓ edge coloring

Let a vertex be v on edge $e = (u,v)$ and an edge ei ($i = 1,2, \ldots, n$) which has the same color of e and $dist(v,ei) = \ell$. For a integer $\ell \geq 1$ and $\beta \geq 0$, CIR_ℓ edge coloring is assigned colors to edges in the same way of ℓ-distance edge coloring, but it is possible that a number of edges ei is equal to or less than β. When e allows some ei by the endpoint v, the number is called the *allowable amount* of e with v (Fig.6).

CIR edge coloring introduced [4] is the method in consideration of the degree of interference into consideration. Although edges in it have the weights in [4], we assume that the all weight is one in this paper. In addition, not only for the ordinary edge coloring and it was extended to ℓ-distance.

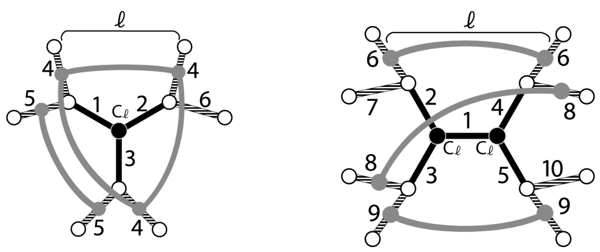
We use the new structures to assume this chromatic number. U_ℓ is a set of edges in S_ℓ such that if a edge $e \in U_\ell$, $dist(e, C_\ell) = \lfloor \frac{\ell}{2} \rfloor$ in S_ℓ. These edges are end of S_ℓ in case of T_d and then the number of edges of U_ℓ,

$$|U_\ell| = \begin{cases} d(d-1)^{\lfloor \frac{\ell}{2} \rfloor} & (\ell : \text{even}) \\ 2(d-1)^{\lfloor \frac{\ell}{2} \rfloor + 1}. & (\ell : \text{odd}) \end{cases} \quad (3)$$



For $\ell = 1$ and $\beta = k$,
k edges can be assigned
the color assigned to e

Fig. 6. CIR_ℓ edge coloring.



$\ell = 2$ (even)

$\chi'_\ell = 9$, c = (2-1)+(3-1) = 3

$\chi'_{CIR,\ell} = 9 - 3 = 6$

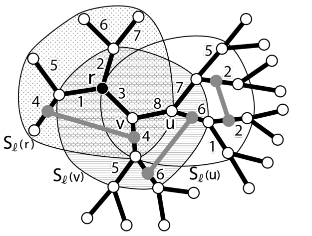
$\ell = 3$ (odd)

$\chi'_\ell = 13$, c = $(2-1) \times 3 = 3$

$\chi'_{CIR,\ell} = 13 - 3 = 10$

U_ℓ is stripe edges and H_ℓ is the gray graph.

Fig. 7. Example of U_ℓ and H_ℓ. ($d = 3$)



The gray edges is of H_ℓ.
The edges of H_ℓ move together S_ℓ,
and corresponding edges are
assigned a same color.

Fig. 8. The way of coloring by CIR_ℓ edge coloring.

Here we define graph H_ℓ with C_ℓ. H_ℓ is called the *allowable graph*. H_ℓ is the graph defined on the ℓ-ball and able to have elements only the edges included in U_ℓ. That is, the vertices of H_ℓ is the edges of S_ℓ. $H_{\ell j}$ ($j = 1, \ldots, n$) is the connected components in H_ℓ, then $|H_\ell|$ will be showed the number of $H_{\ell j}$ n. H_ℓ doesn't contain a trivial graph.

For e, e' and $e'' \in H_\ell$, satisfies the following conditions.

1. If a edge $(e,e') \in H_\ell$, $dist(e,e') = \ell$.
2. If edges (e,e'), $(e',e'') \in H_\ell$, $(e,e'') \in H_\ell$ and $dist(e,e'')$ should be ℓ.
3. If a edge $(e,e') \in H_\ell$, e and e' should be assigned the same color when edges of S_ℓ are assigned colors.

From above condition, all $H_{\ell i}$ is complete graph, and edges of S_ℓ included $H_{\ell j}$ are assigned the same color (Fig.7). Represented by H_ℓ, is where edges we assign the same color to in S_ℓ, we decide the struct of H_ℓ when we assign color to S_ℓ.

The chromatic number is composed of the ℓ-chromatic number and the number of allowable colors that varies depending on β. This result, the chromatic number and β can be seen to correspond in case of ℓ is odd. But if ℓ is even, these are affected by the structure of H_ℓ.

Theorem 3. Let a integer c, $\beta \geq 0$, $\ell \geq 1$ and let T_d be the *d-constant tree* and H_ℓ be the *allowable graph*. Then, the CIR_ℓ-chromatic number $\chi'_{CIR,\ell}(T_d)$ is given by:

$$\chi'_{CIR\ell}(T_d) = \chi'_\ell(T_d) - c \quad (4)$$

such that $0 \leq \beta, c \leq (d-1)^{\lfloor \frac{\ell}{2} \rfloor + 1}$ and

$$\beta = \left\lceil \frac{2|E(H_\ell)|}{d} \right\rceil, c = |V(H_\ell)| - |H_\ell| \quad (\ell : \text{even})$$

$$\beta = c = |E(H_\ell)|. \quad (\ell : \text{odd}) \quad (5)$$

Proof. We have already decided the structure of H_ℓ and get a corresponding c. Then it shows that the chromatic number $\chi'_{CIR\ell}$ is (4) and the allowable amount β is whether it is necessary the number of how much. Finally, we represent a range that could be taken of c and β.

Let assume the coloring in ℓ*-ball* S_ℓ that is the ℓ*-ball* of ℓ-distance. When H_ℓ is already determined, the colors that assign to $E(S_\ell)$ reduce only the elements one drawn for each $H_{\ell i}$. In total, can be reduced the number of colors c,

$$c = \sum_{i=1}^{|H_\ell|} \left\{ |V(H_{\ell j})| - 1 \right\} = |V(H_\ell)| - |H_\ell|. \quad (6)$$

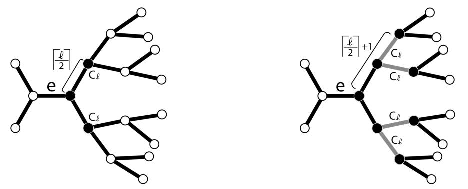
To color the whole T_d, we also use the process of *Th.1*. When S_ℓ is moved, corresponding edges of H_ℓ move together. Therefore, all edges of T_d can be colored with $|E(S_\ell)| - c = \chi'_\ell(T_d) - c$ colors. The uncolored edges in S_ℓ should be assigned in accordance with H_ℓ (Fig.8).

Next we consider that how many β is necessary in above condition. Since required the allowable amount for each edges may be different individually, in this proof, we examine the average per edge. One of edge of H_ℓ shows that corresponding two edges of S_ℓ have one edge of *ei* each other. Then, the allowable amount in S_ℓ is the same to $2|E(H_\ell)|$, and so the average per the endpoint of S_ℓ is $2|E(H_\ell)|/|U_\ell|$. But one edge is included as a endpoint in some ℓ*-ball*. The number of ℓ*-ball* of it is the number of C_ℓ that a distance is $\lfloor \frac{\ell}{2} \rfloor$ from one edge (ℓ is odd, C_ℓ is two. So we select by further one) (Fig.9). Then, for a edge e, the number of S_ℓ that include e as a endpoint is $(d-1)^{\lfloor \frac{\ell}{2} \rfloor}$ (ℓ:even) or $(d-1)^{\lfloor \frac{\ell}{2} \rfloor+1}$ (ℓ:odd). If the average of the allowable amount per edge in T_d is denoted by β, considering β is positive integer, it is

$$\beta = \begin{cases} \left\lceil \dfrac{2|E(H_\ell)|}{d(d-1)^{\lfloor \frac{\ell}{2} \rfloor}} \times (d-1)^{\lfloor \frac{\ell}{2} \rfloor} \right\rceil = \left\lceil \dfrac{2|E(H_\ell)|}{d} \right\rceil & (\ell : \text{even}) \\ \left\lceil \dfrac{2|E(H_\ell)|}{2(d-1)^{\lfloor \frac{\ell}{2} \rfloor+1}} \times (d-1)^{\lfloor \frac{\ell}{2} \rfloor+1} \right\rceil = |E(H_\ell)|. & (\ell : \text{odd}) \end{cases} \quad (7)$$

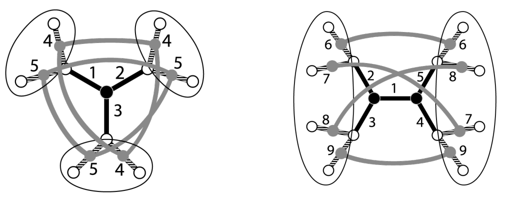
If ℓ is odd, this result is so simple, because H_ℓ has only components of K_2 (K_n is the complete graph of n vertices). Edges of S_ℓ in a same $H_{\ell j}$ should be separated from each other ℓ-1, so a $H_{\ell j}$ only has one edge for each edges that is children of one C_ℓ. Hence, $|V(H_\ell)| = 2|E(H_\ell)|$ and $2|E(H_\ell)| = |H_\ell|$, $c = 2|E(H_\ell)| - |E(H_\ell)| = |E(H_\ell)| = \beta$.

In case of ℓ is even, the above is not true because $H_{\ell j}$ can have one edge for each edges that is children of one C_ℓ's incident edges, C_ℓ's incident edges is d. And H_ℓ may have components in addition to K_2. Then, β and c vary with how we decide the structure of H_ℓ.



$\ell = 2$ (even) $\ell = 3$ (odd)

Fig. 9. The number of ℓ-ball including the edge e.



$\ell = 2$ $\ell = 3$

$\chi'_{CIR\ell} = 9 - 4 = 5$ $\chi'_{CIR\ell} = 13 - 4 = 9$

$d = 3, \beta = (d-1)^2 = 4, c = (d-1)^2 = 4$

Fig. 10. Explanations of the maximum c and β.

Finally, we examine the a range of c and β. Obviously, the lower bound of them is 0, then this coloring is ordinary ℓ-distance edge coloring. To obtain the upper bound, we consider the maximum $|V(H_\ell)|$ and the minimum $|H_\ell|$ at that time. The maximum $|V(H_\ell)| = |U_\ell|$, the maximum $|H_{\ell j}|$ is d (ℓ: even) or 2 (ℓ: odd).(Fig.10) In T_d, all $H_{\ell i}$ can be made up the maximum, then the minimum $|H_\ell| = |U_\ell|/|H_{\ell j}|$. Therefore, considering $|E(K_n)| = n(n-1)/2$ and $|E(H_\ell)| = |H_\ell| \times |E(H_{\ell j})|$, the upper bound of c and β is the following,

if ℓ is even,

$$c = |U_\ell| - \frac{|U_\ell|}{d} = (d-1)(d-1)^{\lfloor \frac{\ell}{2} \rfloor} = (d-1)^{\lfloor \frac{\ell}{2} \rfloor+1}$$
$$\beta = \left\lceil \frac{2|H_\ell| \times |E(K_d)|}{d} \right\rceil = \left\lceil \frac{2(d-1)^{\lfloor \frac{\ell}{2} \rfloor}}{d} \times \frac{d(d-1)}{2} \right\rceil = (d-1)^{\lfloor \frac{\ell}{2} \rfloor+1}, \quad (8)$$

if ℓ is odd,

$$c = |U_\ell| - \frac{|U_\ell|}{2} = (d-1)^{\lfloor \frac{\ell}{2} \rfloor+1}$$
$$\beta = |H_\ell| = \frac{2(d-1)^{\lfloor \frac{\ell}{2} \rfloor+1}}{2} = (d-1)^{\lfloor \frac{\ell}{2} \rfloor+1}. \quad (9)$$

But can be seen to try the calculation, when the above condition, CIR_ℓ edge coloring is ℓ-1-distance. Hence, CIR_ℓ edge coloring can be seen that varying from ℓ-distance and ℓ-1-distance (Fig.10).

IV. CONCLUSION

In this study, we considered the some colorings that assumes in channel assignment in wireless networks. We also provided the chromatic number of them on the graph of tree that the degree is constant.

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