Interpretation of Complex Frequencies in Propagation Problems

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***Abstract*—Derived from Helmholtz wave equation, the dispersion relation describes the temporal and spatial behaviors of a wave in a given electromagnetic medium. In general, this relation can be calculated using either driven mode analysis or eigen mode analysis. In this paper, the two approaches are compared using an example of a periodic leaky-wave antenna. An interpretation for the complex frequencies that arise from the eigen mode approach is given based on the general concept of mapping between two complex planes, namely the complex frequency plane and the complex propagation constant plane. It is shown that the complex frequency dispersion relation gives information on the transient response of the electromagnetic structure under test. Moreover, this paper shows that the results of the two approaches are independent and provide complementary information about the structure under test.**

***Index Terms*—Complex frequency, complex propagation constant, dispersion relation, periodic leaky-wave antenna.**

I. Introduction

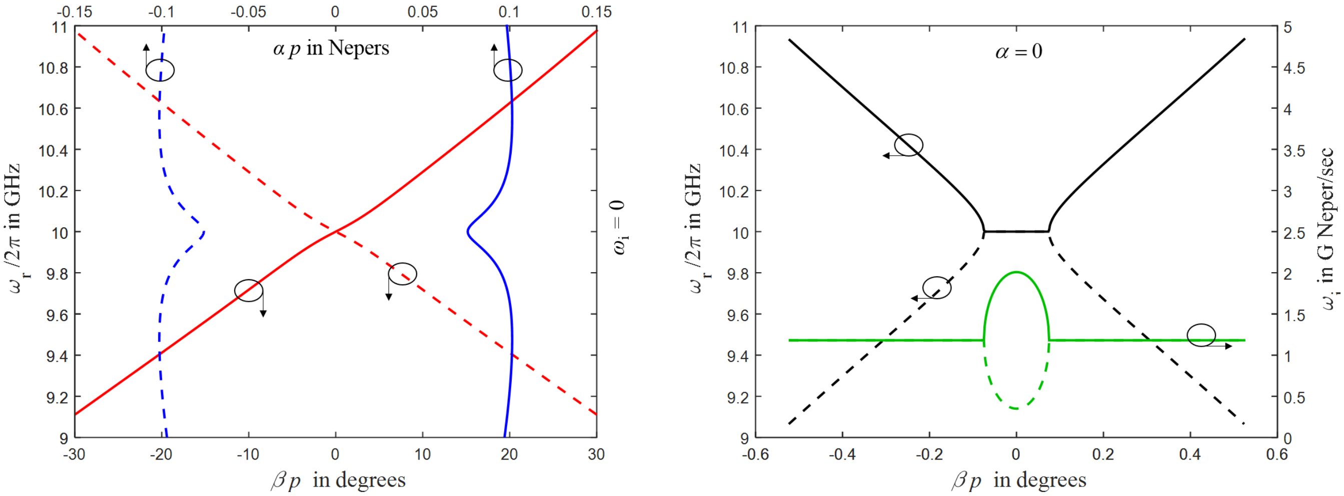
The dispersion relation is a fundamental electromagnetic relation that characterizes propagation in various structures, such as transmission lines, waveguides, photonic crystals, traveling-wave tubes, etc. This relation is commonly used to study the bands of temporal frequencies at which a given structure can operate. It also provides information on the energy dissipation in a given medium, and on the dispersion, hence its name. A common form of visualizing the dispersion relation is the $\omega - \beta$ diagram [1]. On such a diagram, and at a specific frequency of interest (ω), one can read out the value of the propagation constant. A pure imaginary value of the propagation constant (real-valued β) means propagation with no loss, a complex value means propagation with attenuation, and a pure real value (imaginary-valued β) means no propagation. In rare situations, the dispersion relation is plotted with complex temporal frequencies (complex-valued ω). It is not straightforward to see how such information can be used since we know that all sources exciting any electromagnetic structure will be associated with real-valued temporal frequencies. This issue is discussed using an example of a periodic leaky wave antenna P-LWA.

II. Complex Frequency Dispersion Relation of a P-LWA

The design procedure of a P-LWA is given in [2]. The P-LWA antenna radiation pattern is scanned by changing the frequency over the entire space. The design is based on a unit cell that consists of a series and a shunt resonators. For optimum radiation through all scanning angles, the resonant frequency and the quality factor of both resonators have to be equal [2]. A circuit model is used at the first stage of the design, then the lumped components are converted to a distributed microstrip implementation. No matter how accurate the circuit model is, the final microstrip layout must be validated. This is usually done by the use of full wave electromagnetic simulations. There are two approaches for this validation, as shown in the appendix of [2].

The first approach is to cascade a sufficiently large number of its unit cells, then excite the whole structure from both ends with even and odd symmetric fields. This is called drivenmode approach. The even mode analysis gives the parameters of the series resonator, while the odd mode gives those of the shunt resonator. This approach of the unit cell validation has two disadvantages: First, it is sensitive to the number of cells. If the number of cells is too small, then periodicity is not precisely accounted for, and if the number is too large, the parameters calculated become of the whole structure and do not represent the unit cell any more. Second, the computation time is high since the simulated structure consists of multiple replicas of the unit cell.

The second approach for unit cell validation and parameter extraction is based on the eigen mode analysis, where the unit cell is analyzed in a source free situation but with periodic boundary conditions applied at its ends. The advantage of the eigen approach is that only one unit cell needs to be analyzed, so it is time efficient compared to the drivenmode approach. However, in the eigen approach the actual sources are not defined and the wave equation is solved in a source-free situation. The problem is described by an eigen equation where the temporal frequency is the eigen value and the fields are the associated eigen vectors. In such a setup, the solution can yield complex temporal frequencies (eigenvalues). In a



(a) (b)

Fig. 1. Dispersion diagram of the CRLH-PLWA in [2] computed by the two approaches indicated in Section II: (a) Complex propagation constant vs. real temporal frequency (drivenmode computation), (b) Complex temporal frequency vs. pure imaginary propagation constant (eigenmode computation).

practical operation, the antenna is excited by a source with a real-valued temporal frequency. This paper addresses the following questions: What is the physical meaning of the imaginary part of the temporal frequency? And how exactly is it related to the radiation mechanism?

To be more specific, consider the example of the CRLH-PLWA discussed in [2]. The antenna is designed to operate in the bandwidth from 9 to 11 GHz. The radiation pattern is scanned from endfire to backfire through broadside based on the frequency of operation. The broadside frequency, at which the series and shunt resonators are exactly at resonance, is 10 GHz . One can plot the two types of dispersion diagrams mentioned above. These diagrams are shown in Fig. 1.

Figure 1a represents the drivenmode dispersion diagram where the vertical axis is the temporal frequency, and the horizontal axis is the real and imaginary parts of the wave number (α and β). The real part represents the losses along the antenna, including the radiation loss, while the imaginary part represents the phase delay across a unit cell. On the other hand, Fig. 1b shows the dispersion diagram obtained by the eigen mode analysis. The vertical axes represent the real and imaginary parts of the complex temporal frequencies (ω_r and ω_i), while the horizontal axis represents the purely imaginary propagation constant. What does the imaginary part of the frequency mean? And how is it related to the radiation loss? Is there a one to one mapping from Fig. 1a to Fig. 1b? Those questions are answered in the following section.

III. Interpretation of the complex Frequencies in the P-LWA problem

The CRLH-PLWA may be considered as a one-dimensional transmission line along the z direction with cross section in

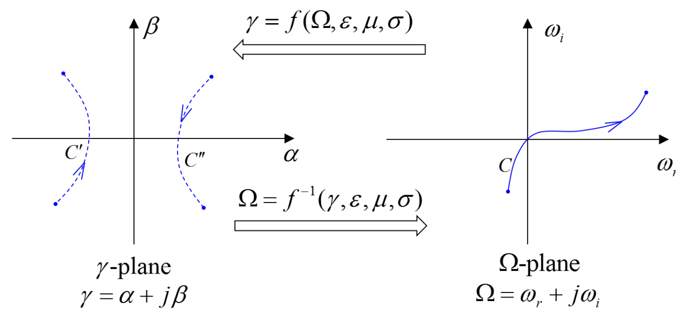


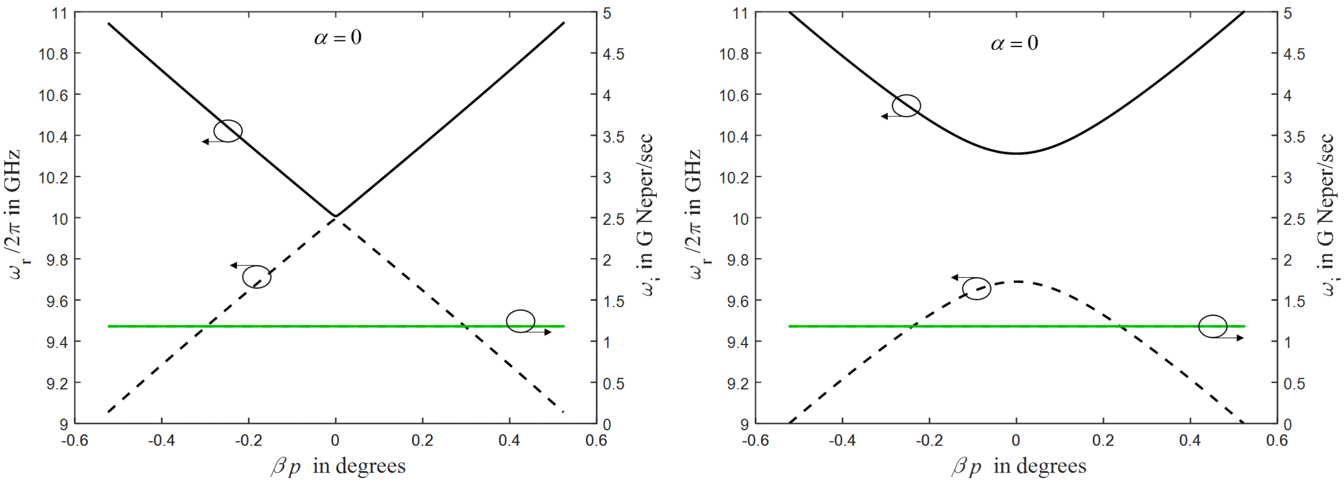
Fig. 2. Mapping between two complex planes through a complex function f. Although f is intended here to be general, the plot shows the case where f is double-valued

the $x-y$ plane. A general form of the electric field along the structure may be written as

$$\mathbf{E}(x,y,z;t)=\mathbf{E}(x,y)e^{\gamma z}e^{j\Omega t}, \tag{1}$$

where $\gamma = \alpha + j\beta$ and $\Omega = \omega_r + j\omega_i$. The expression (1) may be seen as an inverse Laplace transform of a field with single complex frequency, where the Laplace transform is performed over the time variable, t. The form (1) must satisfy the Helmholtz wave equation, from which one can derive the general dispersion relation, i.e. the general relation between γ and Ω.

The dispersion relation may be written in its most general form as $\gamma = f(\Omega;\mu,\epsilon,\sigma)$, where μ is the permeability, ϵ is the permittivity and σ is the conductivity of the medium. In the present example, these parameters may be expressed in terms of the transmission line LCRG parameters. The function f is a complex function relating the propagation constant, γ, and the complex frequency, Ω. The inverse of f is $\Omega = f^{-1}(\gamma;\mu,\epsilon,\sigma)$.



(a) (b)

Fig. 3. Complex frequency (eigenmode) dispersion diagram of the CRLH-PLWA in [2] for (a) the critical damped case for $\beta = 0$, and (b) the underdamped case for $\beta = 0$.

These relations suggest that any contour in the complex Ω plane can be mapped onto the complex γ plane by the means of the function f, and vice versa with f^{-1}. This concept is pictorially illustrated in Fig.2. In a lossless medium, the real axis of the complex Ω plane is mapped onto the imaginary axis of the complex γ plane. In contrast, in a lossy medium, the real axis of the complex Ω plane is mapped via the function f onto a more general contour in the complex γ plane, with a non-zero α. In this case, the wave attenuation in the medium is mathematically represented by α which implies an exponential decay of the field amplitude, $e^{-\alpha z}$. Alternatively, one can map the imaginary axis of the γ plane via f^{-1} onto the Ω plane. Since a LWA is 'lossy', where loss is due to radiation and therefore exists even in the absence of any other kind of loss, the $j\beta$ axis is mapped onto a general contour in the Ω plane, with a non-zero ω_i. According to (1), only ω_i may account for this radiation loss in the structure. However, this loss is represented by an exponential decay of the field amplitude with time in the form $e^{-\omega_i t}$.

Thus, it is important to note that the two diagrams in Fig. 1 represent distinct pieces of information, without any redundancy except for their origin. For instance, all the points on Fig. 1a represent field solutions with complex propagation constants, i.e. $\gamma = \alpha + j\beta$ and $\alpha \neq 0$. On the other hand, $\alpha = 0$ at all points of Fig. 1b. This means that the two diagrams in Fig.1 show the possible solution of the wave equation for different parameter zones, and there is no one to one mapping between them.

A useful perspective to further understand the difference between the two dispersion diagrams in Fig. 1 is to consider Fig. 1a as a steady state representation and Fig. 1b as a transient representation. One may thus consider the complex frequencies as the characteristic roots of the differential equation describing the system. For any system following a second order linear differential equation, the characteristic roots can be classified into one of three categories: over damped, critically damped and under damped [3]. In circuit theory, electric components are considered to be of negligible electric size and therefore circuits do not include any space dependence, i.e. $\beta = 0$. If one extends circuital concepts to the more general case of an electromagnetic medium with large (or unbounded) electric size, the physical significance of the solutions represented in Fig. 1b becomes immediately obvious. For each specific value of β one can solve for the characteristic roots of the system and get either two non-equal real roots (over damped case), or two equal real roots (critically damped case) or two complex conjugate roots (under damped case). Since in (1) we assume a time dependence in the form $e^{j\Omega t}$ instead of $e^{\Omega t}$ as in Laplace transform, the three different cases here correspond to:

- Over damped case: Two different values for ω_i and two equal values for ω_r
- Critically damped case: Two equal values for ω_i and two equal values for ω_r
- Under damped case: Two equal values for ω_i and two different values for ω_r

In Fig. 1b, each value of the horizontal axis (β) represents the boundary conditions applied at the ends of the unit cell. Thus, for each value of β the structure has a different transient behavior characterized by the two complex values of Ω at that specific β. It is also clear that if one changes the geometry of the structure itself, a different solution will arise so as to describe the transient behavior of the new structure.

To illustrate this concept, consider the three different cases given in [2] in the optimization of the radiation of the CRLH-PLWA at broadside. The optimization was done by tuning the transversal symmetry of the unit cell. For each stage of optimization, the dispersion diagram is computed using the eigen mode analysis. Figure 1b represents the symmetric unit cell case, Fig. 3a represents the optimal-asymmetric unit cell case, and Fig. 3b represents the over-asymmetric unit cell case. The difference between the three different figures is significant only around the point $\beta = 0$, which corresponds to the broadside operation of the antenna. At this point, if the characteristic roots of the system (complex frequencies) represent a critically damped case, the radiation at broadside is optimum in the sense that the radiation efficiency does not drop as compared to off-broadside operation. Although the antenna is not practically driven by a complex temporal frequency, the information given in Fig. 3 is useful to characterize the behavior of the system, exactly as the impulse response of a linear time-invariant system is useful to understand the behavior of the system in response to different excitations. In [4], the behavior shown in Fig. 3a was related to the Heaviside condition of distortion-less transmission line, while in this paper, we relate it also to the critical damping nature of the transient response of the antenna.

IV. Conclusion

The drivenmode analysis and eigen mode analysis provide complementary information about an electromagnetic structure. There is no one-to-one correspondence between the dispersion relation curves resulting from each of the two analyzes. In case of a distributed structure, the complex temporal frequency dispersion relation provides information on the nature of the transient behavior of the structure. It turns out that in the particular case of a CRLH-PLWA, optimum radiation at broadside is achieved when the impulse response of the antenna has the form of a critically damped response.

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