Generalized Pascal Matrices and Inverses Using One-to-One Rational Polynomial s-z Transformations

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Abstract: This paper proposes a one-to-one mapping between the coefficients of continuous-time (s-domain) and discrete-time (z-domain) IIR transfer functions such that the s-domain numerator/denominator coefficients can be uniquely mapped to the z-domain numerator/denominator coefficients, and vice versa. The one-to-one mapping provides a firm basis for proving the inverses of the so-called generalized Pascal matrices from various first-order s-z transformations.

1. Introduction

The so-called s-z transformations provide a simple way for analyzing and designing discrete-time (DT) linear systems in the z-domain through utilizing continuous-time (CT) linear systems in the s-domain, and vice versa [1]. For designing infinite-impulse-response (IIR) DT linear filters, it is the most common practice and successful way to convert a CT filter to a DT filter. This is because CT filter design techniques have been highly advanced, thus the existing closed-form solutions and well-documented tables can be adopted directly for designing DT filters. One can design a DT filter with specified design requirements by performing s-z transformations, and the resulting DT filters can preserve the desired characteristics of the original CT filters [2]. On the other hand, CT filters are still important system components and useful in the applications where both CT and DT systems have to co-exist. If the required CT filter is difficult to design in the s-domain, a DT filter can be first designed, and then inverse conversion from z-to-s domain is applied to get a CT filter. Therefore, both s-to-z and z-to-s transformations, i.e., the mutual transformations between s-domain and z-domain are required.

In [1], it is shown that the bilinear (BL) transformation leads to a unique relation between the coefficients of the s-domain and z-domain transfer functions. However, for the IIR case, the mapping between the numerators *only* [3], or the mapping between the denominators *only* does not yield a one-to-one mapping between the coefficients of s-domain and z-domain IIR transfer functions. Consequently, for inverse s-z transformation, the original coefficients cannot be recovered from the transformed coefficients through using the inverse Pascal matrix.

Starting from the general model of the first-order s-z transformations introduced in [3], this paper proposes a one-to-one coefficient mapping between the coefficients of the numerator pair and denominator pair, which has the following advantages.

1) The coefficients of the IIR CT filter $H_{\mathrm{CT}}(s)$ can be uniquely mapped to the coefficients of a DT filter $H(z)$, and vice versa. The coefficients $\{A_n, B_n\}$ are related to $\{a_n, b_n\}$ in a closed-form through using the so-called generalized Pascal matrices and their inverses.

2) The inverses of the generalized Pascal matrices for various first-order s-z transformations can be analytically derived by utilizing the one-to-one mapping. Such proofs and derivations of the inverse generalized Pascal matrices cannot be found in the literature.

2. No One-to-One Mapping Problem

In this section, we first consider why one-to-one mapping between the coefficients of IIR s-domain and z-domain transfer functions are necessary. If not, the inverse transformation (forward or backward s-z transformation) cannot restore the original coefficients. For simplicity, we denote the s-to-z conversion as forward transformation, and z-to-s conversion as backward transformation. To derive correct one-to-one coefficient mapping between the coefficients of s-domain and z-domain IIR transfer functions and correct inverses of the generalized Pascal matrices for various first-order s-z transformations, we need to perform both forward and backward s-z transformations first.

2.1 Forward s-z Transformation

We assume that the s-domain transfer function

$$H_{\mathrm{CT}}(s) = \frac{\sum_{i=0}^{N} A_i s^i}{\sum_{i=0}^{N} B_i s^i} = \frac{N_{\mathrm{CT}}(s)}{D_{\mathrm{CT}}(s)} \tag{1}$$

with

$$N_{\mathrm{CT}}(s) = \sum_{i=0}^{N} A_i s^i, \quad D_{\mathrm{CT}}(s) = \sum_{i=0}^{N} B_i s^i$$

and z-domain transfer function

$$H(z) = \frac{\sum_{k=0}^{N} a_k z^{-k}}{\sum_{k=0}^{N} b_k z^{-k}} = \frac{\sum_{k=0}^{N} a_k z^{N-k}}{\sum_{k=0}^{N} b_k z^{N-k}} = \frac{N(z)}{D(z)} \tag{2}$$

with

$$N(z) = \sum_{k=0}^{N} a_k z^{N-k}, \quad D(z) = \sum_{k=0}^{N} b_k z^{N-k} \tag{3}$$

are related to each other through the first-order s-z transformation

$$s = (-u) \cdot \frac{z-1}{wz-v} \Leftrightarrow z = \frac{u+vs}{u+ws} \tag{4}$$

where u, $v \neq w$ are real constants [3]. Substituting (4) into (1) yields

$$\widehat{H}(z) = \frac{\sum_{i=0}^{N} A_i (-u)^i \left(\frac{z-1}{wz-v}\right)^i}{\sum_{i=0}^{N} B_i (-u)^i \left(\frac{z-1}{wz-v}\right)^i} = \frac{\widehat{N}(z)}{\widehat{D}(z)} \tag{5}$$

where

$$\widehat{N}(z) = \sum_{k=0}^{N} \widehat{a}_k z^{N-k} \tag{6}$$

$$\widehat{a}_k = \sum_{i=0}^{N} M(k,i) \widetilde{A}_i \cdot (-w)^{N-i-k} \tag{7}$$

and

$$M(k,i) = \sum_{n=0}^{k} \binom{N-i}{n} \binom{i}{k-n} v^n w^{k-n}, \quad \widetilde{A}_i = u^i A_i. \tag{8}$$

Using (8), we can prove that the matrix $\mathbf{M}$ takes the form

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \binom{N}{1} v^1 & \cdots & \cdots & \cdots & \binom{N}{1} w^1 \\ \binom{N}{2} v^2 & \cdots & \cdots & \cdots & \binom{N}{2} w^2 \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ \binom{N}{N} v^N & v^{N-1} w^1 & \cdots & v^1 w^{N-1} & \binom{N}{N} w^N \end{bmatrix}. \tag{9}$$

In [3], the matrix $\mathbf{M}$ is called the *generalized Pascal matrix*. For the BL, BD, and parametric BD-BL transformations ($w = -1$), (7) can be rewritten as

$$\widehat{a}_k = \sum_{i=0}^{N} M(k,i) \widetilde{A}_i$$

or we can express the above equation in matrix form as

$$\widehat{\mathbf{a}} = \mathbf{M} \widetilde{\mathbf{A}} \tag{10}$$

with

$$\widehat{\mathbf{a}} = \begin{bmatrix} \widehat{a}_0 \\ \widehat{a}_1 \\ \vdots \\ \widehat{a}_N \end{bmatrix}, \quad \widetilde{\mathbf{A}} = \begin{bmatrix} \widetilde{A}_0 \\ \widetilde{A}_1 \\ \vdots \\ \widetilde{A}_N \end{bmatrix} = \begin{bmatrix} u^0 A_0 \\ u^1 A_1 \\ \vdots \\ u^N A_N \end{bmatrix}. \tag{11}$$

If we just equate the numerator $\widehat{N}(z)$ in (6) to the numerator $N(z)$ in (3) as [3], i.e., $\widehat{a}_k = a_k$, then the numerator coefficient vector $\mathbf{a}$ is related to the vector $\widetilde{\mathbf{A}}$ as

$$\widehat{\mathbf{a}} = \mathbf{a} = \mathbf{M} \widetilde{\mathbf{A}}. \tag{12}$$

We will show later that equating the numerators only will cause no one-to-one coefficient mapping and incorrect inverses of the generalized Pascal matrices.

2.2 Backward s-z Transformation

Conversely, we want to transform the z-domain transfer function

$$H(z) = \frac{\sum_{i=0}^{N} a_i z^{-i}}{\sum_{i=0}^{N} b_i z^{-i}} = \frac{\sum_{i=0}^{N} a_i z^{N-i}}{\sum_{i=0}^{N} b_i z^{N-i}} = \frac{N(z)}{D(z)} \tag{13}$$

to the s-domain transfer function

$$H_{\text{CT}}(s) = \frac{\sum_{k=0}^{N} A_k s^k}{\sum_{k=0}^{N} B_k s^k} = \frac{N_{\text{CT}}(s)}{D_{\text{CT}}(s)} \tag{14}$$

by using the first-order s-z transformation (4). Substituting $z = (u+vs)/(u+ws)$ into (13) yields

$$\widehat{H}_{\text{CT}}(s) = \frac{\sum_{i=0}^{N} a_i \left(\frac{u+vs}{u+ws}\right)^{N-i}}{\sum_{i=0}^{N} b_i \left(\frac{u+vs}{u+ws}\right)^{N-i}} = \frac{\widehat{N}_{\text{CT}}(s)}{\widehat{D}_{\text{CT}}(s)} \tag{15}$$

with

$$\begin{aligned} \widehat{N}_{\text{CT}}(s) &= \sum_{i=0}^{N} a_i (u+vs)^{N-i} (u+ws)^i = \sum_{k=0}^{N} \widehat{A}_k s^k \\ \widehat{A}_k &= u^{N-k} \sum_{i=0}^{N} M(k,i) a_i. \end{aligned} \tag{16}$$

In [3], the numerator $\widehat{N}_{\text{CT}}(s)$ is equated to $N_{\text{CT}}(s)$ in (14), i.e., $\widehat{A}_k = A_k$, which leads to

$$u^{-N} \left(u^k A_k\right) = \sum_{i=0}^{N} M(k,i) a_i, \quad \text{i.e.,} \quad u^{-N} \widetilde{A}_k = \sum_{i=0}^{N} M(k,i) a_i$$

with $\widetilde{A}_k = u^k A_k$. Thus, we have

$$u^{-N} \widetilde{\mathbf{A}} = \mathbf{M} \mathbf{a} \tag{17}$$

i.e.,

$$\mathbf{a} = u^{-N} \left(\mathbf{M}^{-1} \widetilde{\mathbf{A}}\right). \tag{18}$$

For the BL transformation, comparing (12) with (18) leads to

$$\mathbf{M}_{\text{BL}}^{-1} = u^N \mathbf{M}_{\text{BL}} \tag{19}$$

where $\mathbf{M}_{\text{BL}}$ denotes the generalized Pascal matrix for the BL case, and its elements are

$$M_{\text{BL}}(k,i) = \sum_{n=0}^{k} \binom{N-i}{n} \binom{i}{k-n} (-1)^{k-n}. \tag{20}$$

However, the inverse (19) is incorrect, the correct one is

$$\mathbf{M}_{\mathrm{BL}}^{-1} = 2^{-N}\mathbf{M}_{\mathrm{BL}}. \tag{21}$$

The reason for the incorrect relation (19) between the generalized Pascal matrix and its inverse is due to the direct mapping between the numerators *only* or between the denominators *only*. This means that we cannot restore the original s-domain coefficients $\{A_n, B_n\}$ from the z-domain coefficients $\{a_n, b_n\}$, or vice versa, i.e., the coefficient mapping is not one-to-one mapping. Let us see the example ($N = 3$) with s-domain lowpass transfer function

$$H_{\mathrm{CT}}(s) = \frac{5.153 + s^2}{5.153 + 4.344s + 2.781s^2 + 0.929s^3}. \tag{22}$$

For simplicity, we assume $u = 2/T = 1$, which leads to

$$\widetilde{\mathbf{A}} = \mathbf{A} = \begin{bmatrix} 5.153 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \widetilde{\mathbf{B}} = \mathbf{B} = \begin{bmatrix} 5.153 \\ 4.344 \\ 2.781 \\ 0.929 \end{bmatrix}$$

as shown in (11). If (12) is used, then we get

$$\begin{aligned} \begin{bmatrix} \widehat{\mathbf{a}} & \widehat{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix} &= \mathbf{M}_{\mathrm{BL}} \begin{bmatrix} \widetilde{\mathbf{A}} & \widetilde{\mathbf{B}} \end{bmatrix} \\ &= \begin{bmatrix} 6.153 & 13.207 \\ 14.459 & 14.235 \\ 14.459 & 11.121 \\ 6.153 & 2.661 \end{bmatrix} \end{aligned} \tag{23}$$

i.e., the z-domain transfer function is

$$\widehat{H}(z) = \frac{6.153 + 14.459z^{-1} + 14.459z^{-2} + 6.153z^{-3}}{13.207 + 14.235z^{-1} + 11.121z^{-2} + 2.661z^{-3}}.$$

For the backward s-z transformation, if (17) is used as [3], then

$$\begin{aligned} \begin{bmatrix} \widehat{\mathbf{A}} & \widehat{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \end{bmatrix} &= \begin{bmatrix} \widetilde{\mathbf{A}} & \widetilde{\mathbf{B}} \end{bmatrix} \\ = \mathbf{M}_{\mathrm{BL}} \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix} &= \begin{bmatrix} 41.224 & 41.224 \\ 0 & 34.752 \\ 8 & 22.248 \\ 0 & 7.432 \end{bmatrix} \end{aligned}$$

i.e.,

$$\widehat{H}_{\mathrm{CT}}(s) = \frac{41.224 + 8s^2}{41.224 + 34.752s + 22.248s^2 + 7.432s^3}.$$

Clearly, both the numerator and denominator coefficients of $\widehat{H}_{\mathrm{CT}}(s)$ are different from those of $H_{\mathrm{CT}}(s)$ in (22). Therefore, the original s-domain coefficients cannot be restored.

3. Inverse Matrices Using One-to-One Mapping

The incorrect conclusion of the inverse Pascal matrix (19) is caused by equating *only* numerators. Let us go back to (15), where a new s-domain transfer function $\widehat{H}_{\mathrm{CT}}(s)$ is transformed from the z-domain transfer function $H(z)$ in (13). To perform the first-order s-z transformation (4), the constraint

$$s = 0 \Longleftrightarrow z = 1$$

is imposed, i.e, the point $s = 0$ in the s-plane must be transformed into the $z = 1$ in the z-plane [3]. It follows from (15) that

$$\widehat{H}_{\mathrm{CT}}(s)|_{s=0} = \frac{\sum_{i=0}^{N} a_i u^{N-i} \cdot u^i}{\sum_{i=0}^{N} b_i u^{N-i} u^i} = \frac{\sum_{i=0}^{N} a_i u^N}{\sum_{i=0}^{N} b_i u^N} = \frac{\widehat{A}_0}{\widehat{B}_0}. \tag{24}$$

On the other hand, we obtain from (13) that

$$H(z)|_{z=1} = \frac{\sum_{i=0}^{N} a_i}{\sum_{i=0}^{N} b_i}. \tag{25}$$

Comparing (24) with (25) indicates that both the numerator and denominator coefficients of $H(z)$ are scaled by a factor u^N after the backward s-z transformation. However, if we want to derive a one-to-one mapping between the coefficients of the numerators $N_{\mathrm{CT}}(s)$ in (14) and $N(z)$ in (13), the correspondence

$$\begin{aligned} N(z)|_{z=1} = \sum_{i=0}^{N} a_i z^{N-i}|_{z=1} &= \sum_{i=0}^{N} a_i \\ &\Updownarrow \\ N_{\mathrm{CT}}(s)|_{s=0} = \sum_{k=0}^{N} A_k s^k|_{s=0} &= \sum_{k=0}^{N} A_k 0^k = A_0 \end{aligned} \tag{26}$$

must be satisfied. Therefore, we need to divide $\widehat{A}_k$ in (16) by u^N and then set the result to A_k as

$$\frac{\widehat{A}_k}{u^N} = u^{-k} \sum_{i=0}^{N} M(k,i) a_i = A_k. \tag{27}$$

As a result, we have

$$\widetilde{A}_k = u^k A_k = \sum_{i=0}^{N} M(k,i) a_i$$

or in matrix form

$$\widetilde{\mathbf{A}} = \mathbf{M}\mathbf{a}, \quad \text{namely,} \quad \mathbf{a} = \mathbf{M}^{-1}\widetilde{\mathbf{A}}. \tag{28}$$

Next, let us go back to (5), where a new z-domain transfer function $\widehat{H}_{\mathrm{CT}}(z)$ is transformed from the s-domain transfer function $H_{\mathrm{CT}}(s)$ in (1). Since

$$\widehat{H}(z)|_{z=1} = \frac{\sum_{i=0}^{N} u^i A_i \cdot 0^i (v-w)^{N-i}}{\sum_{i=0}^{N} u^i B_i \cdot 0^i (v-w)^{N-i}} = \frac{A_0 (v-w)^N}{B_0 (v-w)^N} \tag{29}$$

and

$$H_{\mathrm{CT}}(s)|_{s=0} = \frac{\sum_{i=0}^{N} A_i 0^i}{\sum_{i=0}^{N} B_i 0^i} = \frac{A_0}{B_0} \tag{30}$$

we know that the numerator and denominator coefficients of $H_{\mathrm{CT}}(s)$ are scaled by a factor $(v-w)^N$ after the forward s-z transformation. Therefore, the coefficients $\widehat{a}_k$ in (7) are related to the coefficients a_k in (2) as

$$\frac{\widehat{a}_k}{(v-w)^N} = a_k. \tag{31}$$

3.1 Bilinear (BL) Transformation

For the BL transformation, $u = \frac{2}{T}$, $v = 1$, $w = -1$, and the generalized Pascal matrix reduces to the so-called *Pascal matrix*, whose elements are defined as (20). As for the inverse Pascal matrix, substituting $v = 1$ and $w = -1$ into (31) yields $a_k = 2^{-N}\widehat{a}_k$, i.e.,

$$\mathbf{a} = 2^{-N}\widehat{\mathbf{a}}. \tag{32}$$

Substituting (10) into (32) results in

$$\mathbf{a} = 2^{-N}\mathbf{M}_{\mathrm{BL}}\widetilde{\mathbf{A}} \tag{33}$$

and comparing (28) and (33) yields the inverse Pascal matrix (21) for the BL case.

3.2 Backward-Difference (BD) Transformation

For the BD transformation, $u = \frac{1}{T}$, $v = 0$, and $w = -1$. Using the general definition (8), we can compute the elements of the generalized Pascal matrix as

$$M_{\mathrm{BD}}(k,i) = \sum_{n=0}^{k} \binom{N-i}{n}\binom{i}{k-n} 0^n (-1)^{k-n} = \binom{i}{k}(-1)^k. \tag{34}$$

For $k > i$, $M_{\mathrm{BD}}(k,i) = 0$, and for $k = i$, $M_{\mathrm{BD}}(k,i) = (-1)^k$. As for the inverse of the generalized Pascal matrix, substituting $v = 0$ and $w = -1$ into (31) yields $\widehat{a}_k = a_k$, i.e.,

$$\mathbf{a} = \widehat{\mathbf{a}}. \tag{35}$$

Substituting (10) into (35) obtains

$$\mathbf{a} = \mathbf{M}_{\mathrm{BD}}\widetilde{\mathbf{A}} \tag{36}$$

and comparing (28) and (36) yields the inverse matrix for the BD case as

$$\mathbf{M}_{\mathrm{BD}}^{-1} = \mathbf{M}_{\mathrm{BD}}. \tag{37}$$

3.3 Forward-Difference (FD) Transformation

For the FD transformation, $u = \frac{1}{T}$, $v = 1$, and $w = 0$. By using the general definition (8), we can express the elements of the generalized Pascal matrix as

$$M_{\mathrm{FD}}(k,i) = \sum_{n=0}^{k} \binom{N-i}{n}\binom{i}{k-n} \cdot 1^n \cdot 0^{k-n} = \binom{N-i}{k}. \tag{38}$$

For $k > N-i$, i.e., $k+i > N$, $M_{\mathrm{FD}}(k,i) = 0$, and for $k+i = N$, $M_{\mathrm{FD}}(k,i) = 1$. As for the inverse matrix, substituting $v = 1$ and $w = 0$ into (31) yields $\widehat{a}_k = a_k$. The $\widehat{a}_k$ in (7) can be further manipulated as

$$\widehat{a}_k = \sum_{i=0}^{N} W(k,i)\widetilde{A}_i \tag{39}$$

with

$$W(k,i) = \binom{i}{N-k}(-1)^{(k+i)-N}. \tag{40}$$

The relation (39) can be expressed in matrix form as

$$\widehat{\mathbf{a}} = \mathbf{a} = \mathbf{W}\widetilde{\mathbf{A}}. \tag{41}$$

By comparing (28) with (41), we obtain the inverse Pascal matrix for the FD case as

$$\mathbf{M}_{\mathrm{FD}}^{-1} = \mathbf{W}. \tag{42}$$

The matrix $\mathbf{W}$ can be easily obtained through flipping the rows and columns of $\mathbf{M}_{\mathrm{FD}}$ and then multiplying the element by $(-1)^{(k+i)-N}$.

3.4 Parametric BD-BL Transformation

For the BD-BL transformation, $u = \frac{1+r}{T}$, $v = r$, and $w = -1$. By using the general definition (8), we can compute the elements of the generalized Pascal matrix as

$$M_{\mathrm{P}}(k,i) = \sum_{n=0}^{k} \binom{N-i}{n}\binom{i}{k-n} r^n (-1)^{k-n}. \tag{43}$$

Also, substituting $v = r$ and $w = -1$ into (31) yields $a_k = (r+1)^{-N}\widehat{a}_k$, i.e.,

$$\mathbf{a} = (r+1)^{-N}\widehat{\mathbf{a}}. \tag{44}$$

Substituting (10) into (44) yields

$$\mathbf{a} = (r+1)^{-N}\mathbf{M}_{\mathrm{P}}\widetilde{\mathbf{A}} \tag{45}$$

and comparing (28) and (45) obtains the inverse matrix for the parametric BD-BL case as

$$\mathbf{M}_{\mathrm{P}}^{-1} = (r+1)^{-N}\mathbf{M}_{\mathrm{P}}. \tag{46}$$

4. Conclusion

In this paper, we have proposed a one-to-one mapping method for transforming the s-domain IIR transfer functions to the z-domain ones or vice versa using the first-order s-z transformations. Based on the one-to-one mapping, we have proved various inverses of the generalized Pascal matrices.

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