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A Novel Pilot Pattern Design Criterion for Compressed Sensing-based Sparse Channel Estimation in OFDM Systems

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***Abstract*—In orthogonal frequency division multiplexing (OFDM) systems, the compressed sensing (CS) technology was proposed for pilot-based sparse channel estimation to improve the bandwidth efficiency and/or estimation performance. The design of pilot patterns plays an important role for CS-based channel estimation. Conventionally, the CS-based pilot pattern design is based on the criterion of minimizing the mutual coherence of the corresponding measurement matrix. In this work, we investigate the relation between a pilot pattern and the corresponding measurement matrix, and derive a new and tighter upper bound of reconstruction error for CS-based channel estimation. According to the upper bound, we propose a new criterion for pilot pattern design in order to achieve better estimation performance. Based on the simulation results, the pilot pattern based on the proposed criterion improves the estimation performance with a gain between 1.5 dB to 2.5 dB in normalized mean square error (NMSE).**

Index Terms*—Compressed sensing (CS); orthogonal frequency division multiplexing (OFDM); sparse channel; channel estimation.

I. Introduction

Orthogonal frequency division multiplexing (OFDM) offers a promising solution for high data rate services and has become the mainstream technology of 4G wireless communications systems. Because of the frequency-selective fading nature in OFDM systems, channel equalization based on channel estimation is essential for data demodulation at the receiver. Particularly, under a time-varying channel, pilot signals are inserted into each OFDM symbol in order to acquire the dynamic channel state information (CSI). Conventionally, multiple subcarriers are reserved for pilot signals transmission and the channel effect is estimated at the receiver based on least squares (LS), minimum mean square error (MMSE) or maximum likelihood (ML) approaches. The allocation of pilot subcarriers is generally equally-spaced (i.e., regular allocation) for uniformly sampling in the frequency domain. Hence, the number of required pilot subcarriers is proportional to the time-domain delay spread of the channel. Under a channel with a very large time delay-spread, numerous pilot subcarriers are required for providing sufficient resolution in the frequency domain, which results in a waste of spectral resource.

In some propagation environments, the multipath channel tends to have a very large delay spread but with a sparse distribution. In other words, the number of channel taps is very large, but the channel is dominated by only a relative few of non-zero taps. For example, both the high-definition television (HDTV) channels and underwater acoustic channels exhibit similar distributions [1], [2]. Under this scenario, the conventional approaches are either quite inefficient or the estimation performance is unacceptable for using a limited number of pilot subcarriers. The compressed sensing (CS, also known as compressive sensing) technology is known to be an effective approach that can reconstruct the original signals from a set of observations significantly fewer than the number traditionally thought necessary [3]-[5]. Under the condition that the system satisfies certain properties, we can acquire the original signals with a high probability by means of some recovery algorithms. The CS technology has been applied to the applications such as pattern recognition, imaging sensors, machine learning, communications, and sensor networks [6]. By utilizing the CS technology for pilot-based sparse channel estimation in OFDM systems, only fewer subcarriers are required for pilot transmission in order to achieve good estimation performance, e.g., [7]-[10].

Unlike the conventional approaches generally using regular pilot patterns, the design of pilot patterns for CS-based channel estimation plays an important role in OFDM systems [11]-[16]. Because the applied pilot pattern (i.e., the locations of the pilot subcarriers) corresponds to the measurement matrix of CS estimation, the goal of pilot pattern design is to make the measurement matrix satisfy certain properties for reconstruction error minimization. In principle, the reconstruction error is determined by the orthogonality of the measurement matrix. Hence, mutual coherence, which is a measure of matrix orthogonality, of the corresponding measurement matrix is commonly used as a design criterion of pilot patterns. Based on the relation between mutual coherence and the upper bounds of reconstruction error proposed in [17], several pilot pattern design methods are proposed to optimize the estimation

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performance. In [11], by the random search approach, a great amount of pilot patterns are randomly generated and the one achieving the minimum mutual coherence in the corresponding measurement matrix is chosen as the optimal pilot patterns. To reduce the searching complexity, several works propose efficient approaches for pilot design based on some optimization algorithms [12]-[16]. A tree-based backward pilot generation scheme is proposed in [12]. In [13], the pilot design algorithm based on the cyclic different set (CDS) is proposed by using a sub-optimal approach. The genetic algorithm (GA) is adopted in the searching process to obtain a sub-optimal pilot pattern efficiently [14], [15]. Similarly, based on the use of constrained cross-entropy optimization, a pilot pattern searching algorithm was proposed for cognitive radio (CR)-OFDM systems in [16]. Note that all the above-mentioned works are based on the design criterion of minimizing the mutual coherence of the measurement matrix for reconstruction error minimization.

In this work, based on the CS-problem for channel estimation in OFDM systems, we investigate the relation between a pilot pattern and the corresponding measurement matrix, and derive a new and tighter upper bound of reconstruction error. According to the upper bound, we propose a new criterion for pilot pattern design in order to obtain better estimation performance. In the simulations, we observe that the pilot based on the proposed design criterion outperforms that based on mutual coherence, resulting in significant improvement in channel estimation performance.

II. Preliminaries

A. System Model

The considered OFDM system comprises N subcarriers, where P out of N subcarriers are employed as pilot subcarriers for channel estimation, and the other $N-P$ subcarriers are used as data subcarriers. By applying an N-point inverse discrete Fourier transform (IDFT) on the frequency-domain signals, we obtain the time-domain OFDM symbol $\mathbf{x} = [x_0,\ x_1,\ \cdots,\ x_{N-1}]^T$ for transmission. Before transmission, a cyclic prefix (CP) with length N_{cp} is inserted to prevent the inter-symbol interference (ISI) caused by channel delay spread. The propagation channel is assumed to be an L-tap multipath fading channel, i.e., the maximum delay spread is L sample intervals, which is smaller than the CP length N_{cp}. The channel impulse response can be represented by a finite impulse response (FIR) filter as

$$h(n) = \sum\nolimits_{\ell=0}^{L-1} h_\ell \delta(n-\ell), n = 0, \cdots, L-1 \qquad (1)$$

where the channel coefficients h_ℓ, for $0 \leq \ell \leq L-1$, are assumed to be complex-Gaussian distributed and time-invariant during an OFDM symbol interval. Note that the channel is assumed to be a sparse channel; that is, the channel is dominated by a few taps with significant path gains and the rest of taps have negligible path gains. For example, if the number of non-zero taps is S, we have $S \ll L$.

After passing through the propagation channel and removing the CP, the received time-domain discrete signal can be expressed as

$$r[i] = \sum\nolimits_{\ell=0}^{L-1} h_\ell x[i-\ell] + v[i],\ \text{for } 0 \leq i \leq N-1. \qquad (2)$$

At the receiver, by applying an N-point discrete Fourier transform (DFT), the frequency-domain signal $\mathbf{R} = [R_0,\ R_1,\ \cdots,\ R_{N-1}]^T$ is obtained. The relation between the receive signal vector $\mathbf{R}$ and the transmit signal $\mathbf{X}$ can be expressed as

$$\mathbf{R} = \mathbf{X}\mathbf{W}\mathbf{h} + \mathbf{v}, \qquad (3)$$

where $\mathbf{X} = \operatorname{diag}(X_0,\ X_1,\ \cdots,\ X_{N-1})$ denotes the frequency domain signals on the N subcarriers, including P pilot subcarriers and $N-P$ data subcarriers; $\mathbf{h} = [h_0,\ h_1,\ \cdots,\ h_{L-1}]^T$ represents the multipath channel response; $\mathbf{v} = [v_0,\ v_1,\ \cdots,\ v_{N-1}]^T$ is the channel noise vector, representing the received complex additive white Gaussian noise (AWGN), which has mean zero and covariance matrix $\sigma_v^2 \mathbf{I}_N$; and $\mathbf{W}$ is an $N \times L$ matrix composed of the first L columns drawn from the full DFT matrix, i.e.,

$$\mathbf{W} = \begin{bmatrix} \omega_{0,0} & \omega_{0,1} & \cdots & \omega_{0,L-1} \\ \omega_{1,0} & \omega_{1,1} & \cdots & \omega_{1,L-1} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{N-1,0} & \omega_{N-1,1} & \cdots & \omega_{N-1,L-1} \end{bmatrix} \qquad (4)$$

where $\omega_{i,\ell} = e^{-j2\pi i\ell/N}$, for $0 \leq i \leq N-1$ and $0 \leq \ell \leq L-1$.

B. CS-based Reconstruction

Consider a linear system defined as

$$\mathbf{r} = \mathbf{\Phi}\mathbf{u} + \mathbf{z}, \qquad (5)$$

where $\mathbf{r} \in \mathbb{R}^K$ is an observation vector, $\mathbf{\Phi} \in \mathbb{R}^{K \times M}$ is the measurement matrix, $\mathbf{u} \in \mathbb{R}^M$ is a vector of the desired unknown parameters, and $\mathbf{z} \in \mathbb{R}^K$ is a noise vector. In general, if the number of unknown parameters in $\mathbf{u}$ is much larger than the number of observations in $\mathbf{r}$ (i.e., $M \gg K$), (5) becomes an underdetermined system and no unique solution of $\mathbf{u}$ can be obtained. However, under the constraint that $\mathbf{u}$ is S-sparse, i.e., at most S of its entries are nonzero, solving (5) to obtain a unique solution of $\mathbf{u}$ is feasible by using the CS techniques. There are several kinds of recovery algorithms that can be used to acquire $\mathbf{u}$ from $\mathbf{r}$, such as the basis pursuit (BP) approach based on l_1-norm minimization [18] and the orthogonal matching pursuit (OMP) scheme based on greedy algorithms [19], [20]. Because of its low complexity, the OMP scheme has become a popular approach for signal reconstruction in CS.

Based on l_1-norm minimization, the estimate of $\mathbf{u}$ is obtained by

$$\begin{aligned} \hat{\mathbf{u}}_1 = \arg\min_{\mathbf{u}} \ & \|\mathbf{u}\|_1 \\ \text{s.t. } & \|\mathbf{\Phi}\mathbf{u} - \mathbf{r}\|_2 \leq \epsilon. \end{aligned} \qquad (6)$$

It is well-known that if the measurement matrix $\mathbf{\Phi}$ satisfies the restricted isometry property (RIP), the reconstruction error $\|\mathbf{u} - \hat{\mathbf{u}}_1\|_2$ is bounded by the noise level ϵ, where $\epsilon \geq \|\mathbf{z}\|_2$

is the maximum norm of noise [17]. In practice, based on the mutual incoherence property (MIP) [21], the assessment of uniqueness of the sparsest solution in (6) is commonly through measuring the mutual coherence of the measurement matrix. The mutual coherence of a matrix $\mathbf{\Phi}$, denoted by $\mathbb{M}(\mathbf{\Phi})$, is defined as the maximum absolute value of the cross-correlations between any two column vectors of $\mathbf{\Phi}$, i.e.,

$$\mathbb{M}(\Phi) = \max_{1\leq m,n\leq M, m\neq n} \left|\mathbf{c}_m^H \mathbf{c}_n\right| \tag{7}$$

where $\mathbf{c}_m$ is the m-th column vector of $\mathbf{\Phi}$. In general, the orthogonality of $\mathbf{\Phi}$ is inversely proportional to the value of $\mathbb{M}(\mathbf{\Phi})$. For an S-sparse vector $\mathbf{u}$, the reconstruction error is upper bounded by a function of $\mathbb{M}(\mathbf{\Phi})$ [17].

C. CS-based Sparse Channel Estimation

In OFDM systems, channel estimation is generally based on the received signals carried on the pilot subcarriers. For the conventional approaches, numerous pilot subcarriers are required for the estimation of a channel with a large delay spread. By extracting the signals received on pilot subcarriers, we have the $P \times 1$ pilot signal vector $\mathbf{R}_\Omega$ represented as

$$\mathbf{R}_\Omega = \mathbf{X}_\Omega \mathbf{W}_\Omega \mathbf{h} + \mathbf{v}_\Omega, \tag{8}$$

where $\mathbf{R}_\Omega = \left[R_{\Omega_0},\, R_{\Omega_1},\, \cdots,\, R_{\Omega_{P-1}}\right]^T$, in which $R_i, i \in \mathbf{\Omega}$ denotes the symbol received on subcarrier i; $\mathbf{\Omega} = \{\Omega_0,\, \Omega_1,\, \cdots,\, \Omega_{P-1}\}$ is the set of indices corresponding to all pilot subcarriers; $\mathbf{X}_\Omega = \mathrm{diag}\left(X_{\Omega_0},\, X_{\Omega_1},\, \cdots,\, X_{\Omega_{P-1}}\right)$ denotes the diagonal matrix corresponding to the transmit pilot symbols; $\mathbf{W}_\Omega$ is composed of P rows by drawing out the i-th rows from $\mathbf{W}$, $\forall i \in \mathbf{\Omega}$; and $\mathbf{v}_\Omega = \left[v_{\Omega_0},\, v_{\Omega_1},\, \cdots,\, v_{\Omega_{P-1}}\right]^T$ is the vector representing noises on pilot subcarriers. Based on (8), the time-domain channel vector $\mathbf{h}$ can be solved if the number of pilot subcarriers is equal to or larger than the channel length, i.e., $P \geq L$.

Because the considered channel is sparse, we can reduce pilot overhead required for channel estimation, i.e., $P \ll L$, by using the CS techniques. Comparing (8) with (5), we observed that the channel vector $\mathbf{h}$ is the desired unknown parameter vector and $\mathbf{X}_\Omega \mathbf{W}_\Omega$ can be regarded as the measurement matrix. If the measurement matrix $\mathbf{X}_\Omega \mathbf{W}_\Omega$ satisfies RIP, the sparse channel vector $\mathbf{h}$ can be obtained by using some recovery algorithms. In this work, we adopt OMP as the recovery algorithm for sparse channel estimation.

III. Proposed Pilot Pattern Design Criterion

A. Conventional Pilot Pattern Design Criterion

Since the upper bound of the reconstruction error is inversely proportional to the mutual coherence $\mathbb{M}(\mathbf{\Phi})$, a common criterion used for the design of the pilot pattern is to choose the one achieving the smallest $\mathbb{M}(\mathbf{\Phi})$ for CS-based channel estimation. For practical applications, numerous pilot patterns are generated/searched and the one with the minimum mutual coherence among them is chosen as the best pilot pattern [11]-[16], i.e.,

$$\mathbf{\Phi}_{opt} = \arg\min_{\mathbf{\Phi}_i} \left\{\mathbb{M}(\mathbf{\Phi}_i)\right\}, \tag{9}$$

where $\mathbf{\Phi}_i$ is the measurement matrix corresponding to the i-th pilot pattern.

B. Properties of the Measurement Matrix

Before introducing the proposed design criterion, we investigate the properties of the measurement matrix in OFDM channel estimation. For OFDM systems, it is generally assumed that every pilot symbol has equal transmit power E, i.e., $|X_i|^2 = X_i^* \cdot X_i = E, \forall i \in \mathbf{\Omega}$, where $*$ denotes the complex-conjugate operation. By normalizing $\mathbf{X}_\Omega$ to have unit l_2-norm, i.e., $\tilde{\mathbf{X}}_\Omega = \mathbf{X}_\Omega \Big/ \sqrt{PE}$, we have (8) rewritten as

$$\tilde{\mathbf{R}} = \mathbf{\Phi}\mathbf{h} + \tilde{\mathbf{v}}, \tag{10}$$

where $\tilde{\mathbf{R}} = \mathbf{R}_\Omega \Big/ \sqrt{PE}$, $\tilde{\mathbf{v}} = \mathbf{v}_\Omega \Big/ \sqrt{PE}$, and $\mathbf{\Phi} = \tilde{\mathbf{X}}_\Omega \mathbf{W}_\Omega$, which is a $P \times L$ matrix. Note that each column of $\mathbf{\Phi}$ still has unit l_2-norm. Since only the pilot signals are used for channel estimation, the matrix subscript Ω is hereinafter omitted. For a specific channel, let $\mathbf{\Delta} = \{\Delta_0,\, \Delta_1,\, \cdots,\, \Delta_{S-1}\}$ be the support of the underlying $\mathbf{h}$ (i.e., the index set of the non-zero elements in $\mathbf{h}$), and $\mathbf{\Psi}$ be the $P \times S$ sub-matrix which consists of the S columns, corresponding to $\mathbf{\Delta}$, drawn out from $\mathbf{\Phi}$, i.e.,

$$\mathbf{\Psi} = \frac{1}{\sqrt{PE}} \begin{bmatrix} X_{\Omega_0} \cdot \omega_{\Omega_0,\Delta_0} & \cdots & X_{\Omega_0} \cdot \omega_{\Omega_0,\Delta_{S-1}} \\ X_{\Omega_1} \cdot \omega_{\Omega_1,\Delta_0} & \cdots & X_{\Omega_1} \cdot \omega_{\Omega_1,\Delta_{S-1}} \\ \vdots & \ddots & \vdots \\ X_{\Omega_{P-1}} \cdot \omega_{\Omega_{P-1},\Delta_0} & \cdots & X_{\Omega_{P-1}} \cdot \omega_{\Omega_{P-1},\Delta_{S-1}} \end{bmatrix} \tag{11}$$

Under the condition that the OMP algorithm recovers the correct support, the reconstruction error is bounded by [17]

$$\left\|\mathbf{h} - \hat{\mathbf{h}}_{OMP}\right\|_2^2 = \left\|\mathbf{\Psi}^\dagger \cdot \tilde{\mathbf{v}}\right\|_2^2 \leq \left\|\mathbf{\Psi}^\dagger\right\|_2^2 \cdot \left\|\tilde{\mathbf{v}}\right\|_2^2. \tag{12}$$

where $\hat{\mathbf{h}}_{OMP}$ is the channel estimate obtained by using the OMP algorithm, and $\mathbf{\Psi}^\dagger$ represents the MoorePenrose pseudo-inverse matrix of $\mathbf{\Psi}$. Note that the l_2-norm $\left\|\mathbf{\Psi}^\dagger\right\|_2$ is equal to the largest singular value of $\mathbf{\Psi}^\dagger$; or equivalently, the inverse of the smallest singular value of $\mathbf{\Psi}$ [22]. Correspondingly, the reconstruction error is bound by

$$\left\|\mathbf{h} - \hat{\mathbf{h}}_{OMP}\right\|_2^2 \leq \frac{\left\|\tilde{\mathbf{v}}\right\|_2^2}{\lambda_{\min}\left(\mathbf{G}_\Psi\right)} \tag{13}$$

where $\mathbf{G}_\Psi = \mathbf{\Psi}^H \mathbf{\Psi}$ is the Gramian matrix of $\mathbf{\Psi}$ with H denoting the Hermitian transpose, and $\lambda_{\min}\left(\mathbf{G}_\Psi\right)$ is the smallest eigenvalue of $\mathbf{G}_\Psi$. Because $\mathbf{\Phi}$ is determined by the applied pilot pattern $\mathbf{\Omega}$, the objective of pilot pattern design is to find $\mathbf{\Omega}$ for the maximization of $\lambda_{\min}\left(\mathbf{G}_\Psi\right)$ corresponding to $\mathbf{\Phi}$.

C. Upper Bound of Reconstruction Error

In the following, we analyze a strict upper bound of $\lambda_{\min}\left(\mathbf{G}_\Psi\right)$, so as to obtain a tighter upper bound of the reconstruction error. Accordingly, we introduce another pilot design criterion based on the proposed upper bound. Considering the measurement matrix $\mathbf{\Phi}$, the corresponding Gramian matrix is

$$\mathbf{G}_\Phi = \mathbf{\Phi}^H \mathbf{\Phi} = \begin{bmatrix} \phi_{1,1} & \phi_{1,2} & \cdots & \phi_{1,L} \\ \phi_{1,2}^* & \phi_{2,2} & \cdots & \phi_{2,L} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{1,L}^* & \phi_{2,L}^* & \cdots & \phi_{L,L} \end{bmatrix} \tag{14}$$

where $\phi_{m,n}$ is the correlation coefficient between the m-th and n-th columns of $\mathbf{\Phi}$, i.e.,

$$\begin{aligned} \phi_{m,n} &= \tfrac{1}{PE} \times \sum_{k=0}^{P-1} \left(X_{\Omega_k}^* \cdot X_{\Omega_k} \right) \left(\omega_{\Omega_k,m}^* \cdot \omega_{\Omega_k,n} \right) \\ &= \sum_{k=0}^{P-1} \omega_{\Omega_k,(n-m)} \Big/ P, \quad \forall\, 0 \le m \le n \le L-1 \end{aligned} \tag{15}$$

where $X_{\Omega_k}^* \cdot X_{\Omega_k} = E$, and $\omega_{\Omega_k,\ell} = \exp(-j2\pi\Omega_k \ell/N)$, for $0 \le \Omega_k \le N-1$ and $0 \le \ell \le L-1$. Note that $\phi_{m,n}$ conforms the property $0 \le |\phi_{m,n}| \le 1$, $\forall\, m, n$, and $\phi_{m,n} = 1$ for $m = n$. Moreover, based on (15), we observe that the correlation coefficient $\phi_{m,n}$ depends only on the difference between the two column indices m and n for a specific pilot pattern $\mathbf{\Omega}$. By defining $\tilde{\phi}_\ell \triangleq \phi_{m,n}$ for $\ell = n - m$, (15) can be rewritten as

$$\mathbf{G}_\Phi = \begin{bmatrix} 1 & \tilde{\phi}_1 & \cdots & \tilde{\phi}_{L-1} \\ \tilde{\phi}_1^* & 1 & \cdots & \tilde{\phi}_{L-2} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\phi}_{L-1}^* & \tilde{\phi}_{L-2}^* & \cdots & 1 \end{bmatrix} \tag{16}$$

Note that the largest absolute value of the off-diagonal elements in $\mathbf{G}_\Phi$ is the mutual coherence $\mathbb{M}(\mathbf{\Phi})$. Since $\mathbf{\Psi}$ is a sub-matrix composed of S columns drawn out from $\mathbf{\Phi}$, the elements of $\mathbf{G}_\Psi$ are the corresponding elements in $\mathbf{G}_\Phi$, i.e.,

$$\mathbf{G}_\Psi = \begin{bmatrix} 1 & \tilde{\phi}_{\Delta_1-\Delta_0} & \cdots & \tilde{\phi}_{\Delta_{S-1}-\Delta_0} \\ \tilde{\phi}_{\Delta_1-\Delta_0}^* & 1 & \cdots & \tilde{\phi}_{\Delta_{S-1}-\Delta_1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\phi}_{\Delta_{S-1}-\Delta_0}^* & \tilde{\phi}_{\Delta_{S-1}-\Delta_1}^* & \cdots & 1 \end{bmatrix} \tag{17}$$

where $\tilde{\phi}_{\Delta_n-\Delta_m}$ is the correlation coefficient between the Δ_m-th and Δ_n-th columns of $\mathbf{\Phi}$. For a specific channel realization, the reconstruction error relies on the minimum eigenvalue of $\mathbf{G}_\Psi$. As addressed in [23], Theorem 1 provides a lower bound of the minimum eigenvalue of a matrix as follows.

Theorem 1: Let $\mathbf{A}$ be an $n \times n$ complex matrix with real eigenvalues $\lambda(\mathbf{A})$, and let

$$a = \mathrm{tr}(\mathbf{A})/n, \quad b^2 = \mathrm{tr}(\mathbf{A}^2)/n - a^2 \tag{18}$$

where $\mathrm{tr}(\cdot)$ is the trace of a matrix. Then

$$a - b \times \sqrt{n-1} \le \lambda_{\min}(\mathbf{A}) \le a - b\big/\sqrt{n-1}, \tag{19}$$

$$a + b\big/\sqrt{n-1} \le \lambda_{\max}(\mathbf{A}) \le a + b \times \sqrt{n-1}. \tag{20}$$

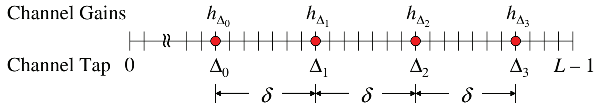

Fig. 1. An example of channel distribution yielding the worse case of $S = 4$.

Equality holds on the left (right) of (19) if and only if equality holds on the left (right) of (20) if and only if the $n-1$ largest (smallest) eigenvalues are equal [23].

Since the Gramian matrix $\mathbf{G}_\Psi$ is an $S \times S$ complex, positive semi-definite matrix, all of its eigenvalues are real and not less than zero. Thus, by Theorem 1, we have

$$a = \mathrm{tr}(\mathbf{G}_\Psi)/S = 1, \quad b = \sqrt{\mathrm{tr}(\mathbf{G}_\Psi^2)/S - 1} \tag{21}$$

and

$$\lambda_{\min}(\mathbf{G}_\Psi) \ge 1 - \sqrt{\left(\frac{\mathrm{tr}(\mathbf{G}_\Psi^2)}{S} - 1 \right)(S-1)} \tag{22}$$

where

$$\mathrm{tr}(\mathbf{G}_\Psi^2) = S + \sum_{\substack{1 \le m,n \le S \\ m \ne n}} g_{m,n} \cdot g_{n,m} = S + 2 \times \sum_{\substack{1 \le m,n \le S \\ m < n}} |g_{m,n}|^2 \tag{23}$$

with $g_{m,n}$ denoting the (m,n)-th element of $\mathbf{G}_\Psi$. Note that the elements of $\mathbf{G}_\Psi$ are specified by the channel support $\mathbf{\Delta}$, or precisely, the index differences between the non-zero taps in $\mathbf{h}$ for a specific pilot pattern. To find the lower bound of $\lambda_{\min}(\mathbf{G}_\Psi)$ for any possible channel realization, we need to find the largest value of $\mathrm{tr}(\mathbf{G}_\Psi^2)$. By ordering the absolute values of correlation coefficients in $\mathbf{G}_\Phi$ in descending order, we have ϕ'_m, for $m = 1, \cdots, L-1$, where $|\phi'_m| \ge |\phi'_{m+1}|$ for $m = 1, \cdots, L-2$. Then, the largest value of $\mathrm{tr}(\mathbf{G}_\Psi^2)$ is achieved when ϕ'_m appears exactly $S - m$ times in $\mathbf{G}_\Psi$, respectively for $m = 1, \cdots, S-1$, i.e., the worst case is

$$\mathbf{G}_\Psi = \begin{bmatrix} 1 & \phi'_1 & \phi'_2 & \cdots & \phi'_{S-1} \\ (\phi'_1)^* & 1 & \phi'_1 & \ddots & \vdots \\ (\phi'_2)^* & (\phi'_1)^* & 1 & \ddots & \phi'_2 \\ \vdots & \ddots & \ddots & \ddots & \phi'_1 \\ (\phi'_{S-1})^* & \cdots & (\phi'_2)^* & (\phi'_1)^* & 1 \end{bmatrix} \tag{24}$$

In other words, the non-zero taps in $\mathbf{h}$ are equally spaced and the corresponding correlation coefficients between them are the largest ones. Hence, we have the worse case of $\mathrm{tr}(\mathbf{G}_\Psi^2)$, which yields the smallest lower bound of $\lambda_{\min}(\mathbf{G}_\Psi)$, as

$$\mathrm{tr}(\mathbf{G}_\Psi^2) \le S + 2 \times \sum\nolimits_{m=1}^{S-1} (S-m) |\phi'_m|^2 \tag{25}$$

In Fig. 1, we show an example of $S = 4$. We assume that the support $\mathbf{\Delta} = \{\Delta_0, \Delta_1, \Delta_2, \Delta_3\}$ of the underlying $\mathbf{h}$ satisfies the following conditions:

1) The non-zero taps in $\mathbf{h}$ are equally spaced, i.e., $\Delta_1 - \Delta_0 = \Delta_2 - \Delta_1 = \Delta_3 - \Delta_2 \triangleq \delta$.

2) According to (16), the correlation coefficients with the first 3 largest absolute values in $\mathbf{G}_\Phi$ are $\phi'_1 = \tilde{\phi}_\delta$, $\phi'_2 = \tilde{\phi}_{2\delta}$, and $\phi'_3 = \tilde{\phi}_{3\delta}$.

Based on condition 1, we have $\Delta_2 - \Delta_0 = \Delta_3 - \Delta_1 = 2\delta$ and $\Delta_3 - \Delta_0 = 3\delta$. Then, according to (17), we obtain the worse case of $\mathbf{G}_\Psi$ as

$$\mathbf{G}_\Psi = \begin{bmatrix} 1 & \phi'_1 & \phi'_2 & \phi'_3 \\ (\phi'_1)^* & 1 & \phi'_1 & \phi'_2 \\ (\phi'_2)^* & (\phi'_1)^* & 1 & \phi'_1 \\ (\phi'_3)^* & (\phi'_2)^* & (\phi'_1)^* & 1 \end{bmatrix}$$

D. Proposed Pilot Pattern Design Criterion

To achieve the best estimation performance, we need to find a pilot pattern that can minimize the value of $\mathrm{tr}(\mathbf{G}_\Psi^2)$ for improving (raising) the lower bound of $\lambda_{\min}(\mathbf{G}_\Psi)$. We propose a novel pilot pattern design criterion based on $\mathrm{tr}(\mathbf{G}_\Psi^2)$. We define the measure metric as

$$\mathbb{W}(\mathbf{\Phi}) = \sum\nolimits_{m=1}^{S-1} (S-m)\left|\phi'_m\right|^2, \tag{26}$$

which can be regarded as a weighted combined mutual coherence of the measurement matrix $\mathbf{\Phi}$. If $\mathbb{W}(\mathbf{\Phi})$ is smaller, a smaller value of $\mathrm{tr}(\mathbf{G}_\Psi^2)$ is obtained. As a result, a higher lower bound of $\lambda_{\min}(\mathbf{G}_\Psi)$ is achieved based on (22) and a smaller reconstruction error is guaranteed based on (13). Therefore, the proposed pilot pattern design criterion is to find the pattern achieves the smallest metric $\mathbb{W}(\mathbf{\Phi})$. The reconstruction error based on the proposed pilot pattern design criterion is given in Theorem 2.

Theorem 2: For the channel estimation problem in OFDM systems as stated in (10), let $\hat{\mathbf{h}}_{OMP}$ be the estimation result of $\mathbf{h}$ via OMP. Then, under the condition that $\hat{\mathbf{h}}_{OMP}$ recovers the correct support, the reconstruction error is bounded by

$$\left\|\mathbf{h} - \hat{\mathbf{h}}_{OMP}\right\|_2^2 \le \frac{\|\tilde{\mathbf{v}}\|_2^2}{1 - \sqrt{2\mathbb{W}(\mathbf{\Phi})(1-1/S)}} \triangleq \frac{\|\tilde{\mathbf{v}}\|_2^2}{\lambda_{\mathbb{W}}} \tag{27}$$

Proof: The proof is omitted here. ■

As reported in [17], in the conventional pilot pattern design criterion based on the mutual coherence, the minimum eigenvalue of $\mathbf{G}_\Psi$ is lower bounded by

$$\lambda_{\min}(\mathbf{G}_\Psi) \ge 1 - \mathbb{M}(\Phi) \times (S-1) \tag{28}$$

Correspondingly, the reconstruction error is upper bounded by

$$\left\|\mathbf{h} - \hat{\mathbf{h}}_{OMP}\right\|_2^2 \le \frac{\|\tilde{\mathbf{v}}\|_2^2}{1 - \mathbb{M}(\Phi) \times (S-1)} \triangleq \frac{\|\tilde{\mathbf{v}}\|_2^2}{\lambda_{\mathbb{M}}} \tag{29}$$

Compared the proposed pilot pattern design criterion $\mathbb{W}(\mathbf{\Phi})$ with the conventional one, we have the following theorem.

Theorem 3: For the channel estimation problem in OFDM systems as stated in (10), let $\hat{\mathbf{h}}_{OMP}$ be the estimation result of $\mathbf{h}$ via OMP. Then, under the condition that $\hat{\mathbf{h}}_{OMP}$ recovers the correct support, the upper bound of reconstruction error based on $\mathbb{W}(\mathbf{\Phi})$ is tighter than that based on $\mathbb{M}(\mathbf{\Phi})$; that is,

$$\left\|\mathbf{h} - \hat{\mathbf{h}}_{OMP}\right\|_2^2 \le \frac{\|\tilde{\mathbf{v}}\|_2^2}{\lambda_{\mathbb{W}}} \le \frac{\|\tilde{\mathbf{v}}\|_2^2}{\lambda_{\mathbb{M}}} \tag{30}$$

TABLE I

STATISTICS OF THE GENERATED PILOT PATTERN FOR THE PILOT DESIGN SCENARIOS A, B AND C.

Scenarios	A	B	C
Number of randomly generated pilot patterns	1,000	100,000	500,000
Number of evaluations	500		
Number of evaluations choosing different patterns	201 (40.2%)	149 (29.8%)	91 (18.2%)

TABLE II

RANGES OF THE METRICS OF THE MEASUREMENT MATRICES BASED ON THE CONVENTIONAL AND PROPOSED CRITERIA.

Criteria	Conventional	Proposed	
	$\mathbb{M}(\mathbf{\Phi})$	$\mathbb{W}(\mathbf{\Phi})$	Corresponding $\mathbb{M}(\mathbf{\Phi})$
Scenario A	(0.274, 0.301)	(0.678, 0.849)	(0.274, 0.330)
Scenario B	(0.250, 0.264)	(0.570, 0.625)	(0.251, 0.278)
Scenario C	(0.239, 0.250)	(0.551, 0.615)	(0.243, 0.278)

where $\lambda_{\mathbb{W}}$ and $\lambda_{\mathbb{M}}$ are respectively defined in (27) and (29).

Proof: The proof is omitted here. ■

IV. PERFORMANCE EVALUATION

In the simulation, we assume that the number of subcarriers in the OFDM system is $N = 512$ and $P = 25$ subcarriers are used for the transmission of pilot symbols. The sparse multipath channel $\mathbf{h}$ has the maximum delay spread $L = 50$ taps and the number of non-zero channel taps is $S = 5$, where the first channel tap h_0 is always non-zero and the other 4 non-zero taps are randomly distributed within the successive taps. The power delay profile of $\mathbf{h}$ follows the exponent decay, i.e., $|h_l|^2 = \exp(-l/L)$ for $l = 0, \cdots, L-1$ [11]. In the signal reconstruction, the OMP scheme is adopted as the recovery algorithm and the stopping condition is based on the known sparsity S, i.e., the OMP algorithm performs S iterations.

As shown in Table I, we consider three scenarios, where the numbers of randomly generated pilot patterns are K = 1,000, 100,000 and 500,000 respectively for Scenario A, Scenario B and Scenario C. Then, the optimal pilot pattern is chosen based on either the conventional criterion $\mathbb{M}(\mathbf{\Phi})$ or the proposed criterion $\mathbb{W}(\mathbf{\Phi})$. We independently evaluate each scenario 500 times to obtain the simulation statistics. Among the 500 times of independent evaluation, different design criteria lead to different optimal patterns with a high probability, i.e., 40.2%, 29.8% and 18.2% respectively for the three scenarios. To compare the estimation performance achieved by the two design criteria, we focus only on the cases when different optimal patterns are chosen by the two criteria. The ranges of the metrics $\mathbb{M}(\mathbf{\Phi})$ or $\mathbb{W}(\mathbf{\Phi})$ corresponding to the optimal pilot patterns are also shown in Table II. For the proposed design criterion, the range of the corresponding mutual coherence $\mathbb{M}(\mathbf{\Phi})$ is also shown as a reference. When different optimal patterns are chosen by the two criteria, the proposed criterion yields the selection of a measurement matrix with a larger mutual coherence.

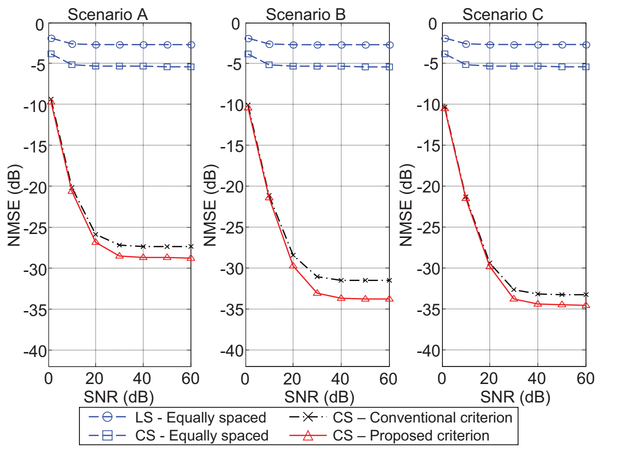

Fig. 2. Comparison of estimation accuracy based on the stopping condition of known sparsity S.

We compare in Fig. 2 the estimation accuracy, measured by normalized mean square error (NMSE), of different design criteria based on the stopping condition of performing the OMP algorithm S iterations. The NMSE is defined as $\left\|\mathbf{h} - \hat{\mathbf{h}}_{OMP}\right\|_2^2 \Big/ \|\mathbf{h}\|_2^2$ for each channel realization, by averaging over the results of 50,000 random channel realizations. In addition to the CS approach based on the conventional and proposed design criteria, we provide the performance of the least square (LS) approach and CS approach by using an equally spaced pilot pattern. For the LS approach, the number of pilot subcarriers $P = 25$ is far less than the minimum required one. As a result, the estimation performance is quite bad. For the CS approach with an equally spaced pilot pattern, the measurement matrix does not have good properties and the corresponding metrics are $\mathbb{M}(\mathbf{\Phi}) = 0.9049$ and $\mathbb{W}(\mathbf{\Phi}) = 5.529$. Hence, the estimation accuracy is still quite bad. However, by using non-uniform pilot pattern, the CS-based performance is greatly improved. Comparing the two criteria, the performance of the proposed criterion based on metric $\mathbb{W}(\mathbf{\Phi})$ is far better the conventional one based on $\mathbb{M}(\mathbf{\Phi})$ with an improvement in NMSE between 1.5 dB to 2.5 dB.

V. Conclusion

In this work, we have proposed a new pilot pattern design criterion for CS-based channel estimation in OFDM systems over a sparse channel. The design criterion is based on a new evaluation metric corresponding to multiple column-correlation coefficients in the measurement matrix. Moreover, an upper bound of reconstruction error corresponding to the proposed evaluation metric is derived. According to the simulation results based on the random search approach, the proposed design criterion leads to an optimal pilot pattern different to that obtained by using the conventional criterion with a high probability. Correspondingly, the pilot pattern based on the proposed criterion outperforms that based on the conventional criterion in the estimation performance, and an improvement between 1.5 dB to 2.5 dB in NMSE can be achieved. The proposed design criterion can be applied to not only the random search approach, but also to other pilot search approaches.

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