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A Linear Time Algorithm for Tri-connectivity Augmentation of Bi-connected Graphs with Upper Bounds on Vertex-Degree Increase

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Abstract: The 3-vertex-connectivity augmentation problem of a graph with degree constraints, 3VCA-DC, is defined as follows: "Given an undirected graph $G = (V, E)$, and an upper bound $b(v) \in Z^+ \cup \{\infty\}$ on vertex-degree increase for each $v \in V$, find a smallest set E' of edges such that $(V, E \cup E')$ is 3-vertex-connected and such that vertex-degree increase of each $v \in V$ by the addition of E' to G is at most $b(v)$, where Z^+ is the set of nonnegative integers." In this paper we show that checking the existence of a feasible solution and finding an optimum solution to 3VCA-DC for any bi-connected graph G can be done in $O(|V| + |E|)$ time.

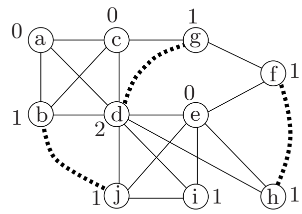

Figure 1. An instance, $G = (V, E)$ and $b : V \to Z^+ \cup \{\infty\}$, of 3VCA-DC, where G is bi-connected and the number beside each vertex $v \in V$ denotes $b(v)$. The edge set $\{(\mathrm{b}, \mathrm{j}), (\mathrm{d}, \mathrm{g}), (\mathrm{f}, \mathrm{h})\}$ shown by bold dotted lines is an optimum solution. Notice that if $b(\mathrm{b}) = 0$ then there is no feasible solution since $b(\{\mathrm{a}, \mathrm{b}\}) = b(\mathrm{a}) + b(\mathrm{b}) = 0$.

1. Introduction

Given an undirected graph $G = (V, E)$ and a subset $S \subseteq V$, G is said to be *k-vertex-connected* (or *k-connected*) *with respect to* S if and only if G has at least k internally-disjoint paths between any pair of vertices in S, where if $S = V$ then "with respect to S" is neglected.

The k-vertex-connectivity augmentation problem for a specified set of vertices of a graph with degree constraints, kVCA-SV-DC, is defined as follows: "Given a positive integer k, an undirected graph $G = (V, E)$, a specified set $S \subseteq V$ and an upper bound $b(v) \in Z^+ \cup \{\infty\}$ on vertex-degree increase for each $v \in V$, find a smallest set E' of edges such that $G + E'$ is k-connected with respect to S and such that vertex-degree increase of each $v \in V$ by the addition of E' to G is at most $b(v)$, where $G + E' = (V, E \cup E')$ and Z^+ is the set of nonnegative integers." We call any set F of edges a *solution to* kVCA-SV-DC if $G + F$ is k-connected with respect to S, and we say that any set F' of edges is *feasible* (or more precisely *b-feasible*) if F' includes at most $b(v)$ edges incident to v for each $v \in V$. Any feasible solution of minimum cardinality is called an *optimum solution to* kVCA-SV-DC. kVCA-SV-DC has application to designing communication networks. kVCA-SV-DC with $S = V$ is denoted as kVCA-DC. kVCA-SV-DC with $b(v) = \infty$ for all $v \in V$ is denoted as kVCA-SV. kVCA-SV with $S = V$ is denoted as kVCA. Figure 1 shows an instance of 3VCA-DC.

We summarize known results on kVCA-SV-DC. For 2VCA, a polynomial time algorithm was proposed by Eswaran and Tarjan [1], and a linear time algorithm was proposed Rosenthal and Goldner [11] and Hsu and Ramachandran [5]. For 3VCA, a polynomial time algorithm was devised by Watanabe and Nakamura [14], and a linear time algorithm was proposed by Hsu and Ramachandran [4]. For 4VCA, Hsu [3] devised a polynomial time algorithm. For any fixed k, Jackson and Jordán [6] first devised a polynomial time algorithm for kVCA. Concerning kVCA-SV, it was shown that kVCA-SV can be solved in linear time by reducing it to kVCA by Watanabe, Higashi, and Nakamura [13] if $k = 2$ and by Mashima and Watanabe [10] if $k = 3$. For 2VCA-SV-DC, we proposed a linear time algorithm [8].

In this paper we show that checking the existence of a feasible solution and finding an optimum solution to 3VCA-DC for any bi-connected graph G can be done in $O(|V| + |E|)$ time. All proofs are omitted due to shortage of space. The full version of the paper appears in [9].

2. Preliminaries

2.1 Basic definitions

An *undirected graph* $G = (V(G), E(G))$ consists of a finite and nonempty set $V(G)$ of vertices and a finite set $E(G)$ of undirected edges. $V(G)$ and $E(G)$ are often denoted as V and E, respectively. We assume that graphs have neither multiple edges nor self-loops unless otherwise stated. An edge whose endvertices are u and v is denoted by (u, v). The *degree* of a vertex v in G, $d_G(v)$, is the number of edges incident to v in G. For a set E' of edges with $E' \cap E = \emptyset$, $G + E'$ denotes the graph $(V, E \cup E')$. For a set $X \subseteq V \cup E$, $G - X$ denotes the graph obtained from G by deleting X, where any edge incident to $v \in X$ is also removed. A *subgraph* of G is any graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $X \subseteq V$, let $\Gamma(X; G) = \{v \in V - X \mid (u, v) \in E(G)$ for some $u \in X\}$. If G is clear from the context, $\Gamma(X; G)$ may be denoted as $\Gamma(X)$. For any function $f : V \to Z^+ \cup \{\infty\}$ and any set $X \subseteq V$, we use the notation $f(X) = \sum_{v \in X} f(v)$.

A *path between* u *and* v, or a (u,v)*-path*, is an alternating sequence of vertices and edges $u = v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n = v$ ($n \geq 0$) such that if $n \geq 1$ then $v_0, \ldots, v_n$ are all distinct and $e_i = (v_{i-1}, v_i)$ for each i, $1 \leq i \leq n$. If $n \geq 2$ then the vertices $v_1, v_2, \cdots, v_{n-1}$ are called the *inner vertices* of the path. A set of paths is said to be *internally-disjoint* if no two of them have an inner vertex in common. Any maximal connected subgraph of G is called a *connected component* or simply a *component* of G. A *tree* is an acyclic connected undirected graph. A *leaf* of a tree is a vertex with only one edge incident to it.

Let G be a connected graph. Any subset R of V is called a *separator of* G if and only if $G - R$ is disconnected. Any separator of minimum cardinality is called a *minimum separator of* G. The *vertex-connectivity* $\kappa(G)$ *of* G is the cardinality of any minimum separator of G, where if G is a complete graph then $\kappa(G)$ is defined to be $|V| - 1$. If $\kappa(G) \geq k$ for a positive integer k then G is said to be k*-vertex-connected* (or k*-connected*); in particular, if $\kappa(G) \geq 2$ ($\kappa(G) \geq 3$) then G is said to be *bi-connected* (*tri-connected*, respectively). Note that $\kappa(G) \geq k$ holds if and only if G has at least k internally-disjoint paths between any pair of vertices in V.

Let G be a bi-connected graph. For any pair of vertices $u, v \in V$, $\{u, v\}$ is called a *separation pair of* G if and only if $\{u, v\}$ is a separator of G. Let $\mathrm{SP}(G)$ denote the class of all separation pairs of G, where the part "(G)" is often omitted for simplicity unless any confusion arises. Let $K \in \mathrm{SP}(G)$. Any component of $G - K$ is called a K*-component* of G. The *separating degree* of K in G is the number of K-components of G, and it is denoted by $d_G(K)$ or simply $d(K)$. In this paper, we often identify a K-component (or a component) with its vertex set.

2.2 Tri-components and 3-block graph

Let G be a bi-connected graph. *Tri-connected components* (*tri-components* for short) [2, 12] of G are constructed from G by the two operations "split" and "merge". Tri-components can be partitions into three classes: tri-connected graphs, polygons and bonds, where a *bond* is a graph consisting of two vertices and at least three multiple edges between them. For the details of tri-components, refer to [2, 12]. Figure 2 shows tri-components of the graph G in Fig. 1. Any separation pair shared by at least two tri-components, each being either a tri-connected graph or a polygon, is called a separation pair of *adjacent type*, and any separation pair consisting of nonadjacent vertices in a polygon is called a separation pair of *nonadjacent type*. Let $\mathrm{SP_A}(G) = \{K \in \mathrm{SP}(G) \mid K$ is of adjacent type$\}$, where the part "(G)" is often omitted for simplicity unless any confusion arises.

The 3-block graph [4], 3-blk(G), of a bi-connected graph G is constructed as follows. The vertex set of 3-blk(G) consists of three kinds of vertices called σ*-vertices*, π*-vertices* or β*-vertices*: we create a σ*-vertex* for each separation pair of adjacent type, a π*-vertex* for each polygon, and a β*-vertex* for each tri-component that is a tri-connected graph. Then distinct vertices u, v created above are connected by an edge if and only if either (1) or (2) holds: (1) $\{u, v\}$ is a pair of a

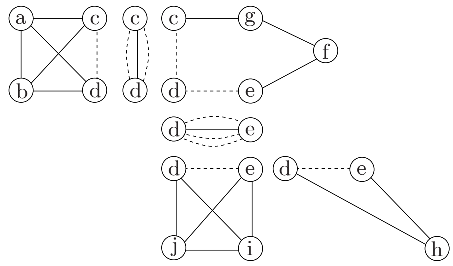

Figure 2. Tri-components of the graph G in Fig. 1. Edges represented as broken lines are virtual edges.

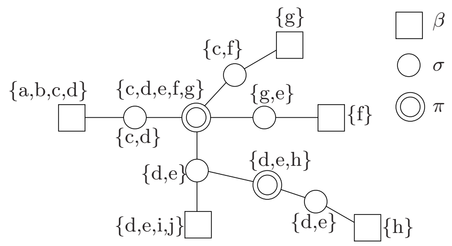

Figure 3. 3-blk(G) of the graph G in Fig. 1.

σ-vertex and a β-vertex, and the separation pair represented by the σ-vertex is contained in the tri-connected graph represented by the β-vertex; (2) $\{u, v\}$ is a pair of a σ-vertex and a π-vertex, and the separation pair represented by the σ-vertex is contained in the polygon represented by the π-vertex. Moreover, for each each vertex $w \in V(G)$ with $d_G(w) = 2$, do the following: create a β-vertex u for $\{w\}$, and a σ-vertex v for $\Gamma(w; G)$; connect u and v by an edge; connect by an edge the vertex v and the π-vertex representing the polygon to which w belongs. The resulting graph is 3-blk(G). 3-blk(G) is a tree and can be constructed in $O(|V| + |E|)$ time by using the algorithm in [2] for finding tri-components of G. Figure 3 shows 3-blk(G) of the graph G in Fig. 1.

Let $v \in V$(3-blk(G)). The degree of v in 3-blk(G) is denoted by $d_{\text{3-blk}(G)}(v)$ or simply $d(v)$ unless any confusion arises. If $d(v) = 1$ then v is called a *leaf* of 3-blk(G).

Any subset $K \subseteq V$ represented by a σ-vertex of 3-blk(G) is called a σ*-pair of* G. Any graph represented by a β-vertex of 3-blk(G) is called a *3-block of* G. Any 3-block B of G is called a *singleton 3-block* if B consists of a single vertex of G; otherwise B is a *tri-connected 3-block*. For any 3-block B of G, let $d_G(B)$ or simply $d(B)$ denote the number of separation pairs $K \in \mathrm{SP}(G)$ with $K \subseteq V(B)$, where we set $d_G(B) = 1$ for any singleton 3-block B. For any polygon P of G, let $d_G(P)$ or simply $d(P)$ denote the sum of the number of separation pairs $K \in \mathrm{SP_A}(G)$ with $K \subseteq V(P)$ and the number of singleton 3-blocks included in P. For any $v \in V$(3-blk(G)), let K_v, B_v or P_v denote the σ-pair, the 3-block or the polygon represented by v, respectively. Note that $d(v) = d_G(B_v)$ for any β-vertex v and that $d(u) = d_G(P_u)$ for any π-vertex u.

Any separation pair K of G is said to be a *primary separation pair* if either $d(K) \geq 3$ or $K \subseteq V(B)$ for some 3-block B with $d(B) \geq 3$. For any $v \in V$ with $d(v) = 2$, let $P_G(v)$ or

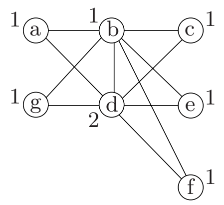

Figure 4. An instance for which there is no feasible solution to 3VCA-DC, where $l(G) = d(G) = 5$ and the condition (2) does not hold. $\{\mathrm{b}, \mathrm{d}\}$ is the separation pair K with $d(K) = d(G)$. Since $b(V-K) = 5 < 8 = 2(d(G)-1)$, there is no feasible solution.

simply $P(v)$ denote the polygon of G containing v. Let T_G or simply T denote the set $\{v \in V \mid d(v) = 2, d(P(v)) \geq 3\}$.

2.3 Lower bounds

Let G be a bi-connected graph with $|V| \geq 4$. Let v be any leaf of 3-blk(G) and let u be the σ-vertex adjacent to v in 3-blk(G). Then the subset $V(B_v) - K_u \subseteq V$ is called a *leaf of* G or *the leaf of* G *represented by* v. Let $\mathcal{L}(G)$ denote the class of all leaves of G, and let $l(G)$ denote the number of leaves of G (or equivalently the number of leaves of 3-blk(G)). Note that $X \cap X' = \emptyset$ for any distinct $X, X' \in \mathcal{L}(G)$. Let $d(G)$ denote the maximum separating degree of all separation pairs of G (or equivalently the maximum degree of all σ-vertices in 3-blk(G)). For G in Fig. 1, $\mathcal{L}(G) = \{\{\mathrm{a}, \mathrm{b}\}, \{\mathrm{f}\}, \{\mathrm{g}\}, \{\mathrm{h}\}, \{\mathrm{i}, \mathrm{j}\}\}$, $l(G) = 5$, and $d(G) = 3$.

Lemma 1: For any feasible solution F to 3VCA-DC for G and b, $|F| \geq \max\{d(G) - 1, \lceil l(G)/2 \rceil\}$. ■

Lemma 2: If either $(d(G)-1 > \lceil l(G)/2 \rceil)$ or $(d(G)-1 = \lceil l(G)/2 \rceil$ and $l(G)$ is odd) then there is the unique σ-vertex $v \in V(\text{3-blk}(G))$ with $d(v) = d(G)$; furthermore, K_v is the unique separation pair $K \in \mathrm{SP}(G)$ with $d(K) = d(G)$. ■

3. The Existence Condition for Feasible Solutions

In this section we show a necessary and sufficient condition for the existence of a feasible solution to 3VCA-DC for any biconnected graph G with $|V| \geq 4$.

Theorem 3: There is a feasible solution to 3VCA-DC for a bi-connected graph G with degree constraints by b if and only if the following (1)–(3) hold.

(1) $b(X) \geq 1$ for any leaf $X \in \mathcal{L}(G)$.

(2) If either $(d(G)-1 > \lceil l(G)/2 \rceil)$ or $(d(G)-1 = \lceil l(G)/2 \rceil$ and $l(G)$ is odd) then $b(V-K) \geq 2(d(G)-1)$ for the separation pair K with $d(K) = d(G)$. (See Fig. 4.)

(3) If $(d(G) - 1 < \lceil l(G)/2 \rceil$ and $l(G)$ is odd) then the following (a)–(c) hold.

(a) $b(V) \geq l(G) + 1$.

(b) If no polygon P with $d(P) \geq 3$ exists and one vertex $u \in V$ is shared by all primary separation pairs of G, then $b(V - \{u\}) \geq l(G) + 1$. (See Fig. 5.)

(c) If $l(G) = 3$, there is a polygon P with $d(P) = 3$, and P includes at least one singleton 3-block, then $b(V-T) \geq l(G) + 1 - |T|$. (See Fig. 6.) ■

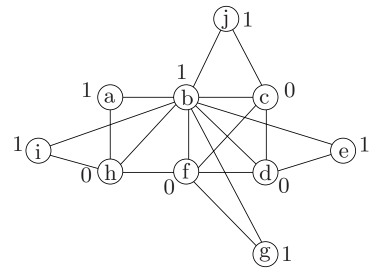

Figure 5. An instance for which there is no feasible solution to 3VCA-DC, where $l(G) = 5$, $d(G) = 3$, and the condition (3)(b) does not hold. Vertex b is shared by all members of $\{K \in \mathrm{SP}(G) \mid d(K) \geq 3\} = \{\{\mathrm{b}, \mathrm{f}\}, \{\mathrm{b}, \mathrm{h}\}\}$ and $\{K \in \mathrm{SP}(G) \mid K \subseteq V(B)$ for some 3-block B with $d(B) \geq 3\ \} = \{\{\mathrm{b}, \mathrm{c}\}, \{\mathrm{b}, \mathrm{d}\}, \{\mathrm{b}, \mathrm{f}\}\}$. Since $b(V - \{\mathrm{b}\}) = 5 < 6 = l(G) + 1$, there is no feasible solution.

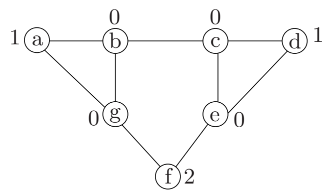

Figure 6. An instance for which there is no feasible solution to 3VCA-DC, where $l(G) = 3$, $d(G) = 2$, and the condition (3)(c) does not hold. Polygon P with $V(P) = \{\mathrm{b}, \mathrm{c}, \mathrm{e}, \mathrm{f}, \mathrm{g}\}$ has $d(P) = 3$, and $T = \{v \in V(P) \mid d_G(v) = 2\} = \{\mathrm{f}\}$. Since $b(V-T) = 2 < 3 = 4 - |T|$, there is no feasible solution.

We call the set of the conditions (1)–(3) given in Theorem 3 *the existence condition for feasible solutions*. Note that, in the condition (2), K is unique by Lemma 2.

Given G and b, we can check whether or not the existence condition for feasible solutions holds in $O(|V| + |E|)$ time. Hence we have the following theorem.

Theorem 4: Checking the existence of a feasible solution to 3VCA-DC for any bi-connected graph G can be done in $O(|V| + |E|)$ time. ■

4. A Linear Time Algorithm

In this section, we assume that a bi-connected graph G with degree constraints by b satisfies the existence condition for feasible solutions. We partition the problem into the following three cases:

Case 1: $(d(G) - 1 > \lceil l(G)/2 \rceil)$ or $(d(G) - 1 = \lceil l(G)/2 \rceil$ and $l(G)$ is odd);

Case 2: $(d(G) - 1 \leq \lceil l(G)/2 \rceil$ and $l(G)$ is even);

Case 3: $(d(G) - 1 < \lceil l(G)/2 \rceil$ and $l(G)$ is odd).

And, we present three linear time algorithms, Algorithm 1 for Case 1, Algorithm 2 for Case 2, Algorithm 3 for Case 3, for finding an optimum solution. Our algorithm *Solve_3VCA-DC_aug2to3* for finding an optimum solution to

3VCA-DC for any bi-connected graph G consisits of applying one of three algorithms according to the case to which the problem belongs.

4.1 Algorithm 1 and Algorithm 2

The following Algorithms 1 and 2 solve Cases 1 and 2, respectively. Algorithm 1 is based on the algorithm in [7], and Algorithm 2 uses an algorithm for 3VCA.

Algorithm 1 /* for Case 1 */

1. Find the separation pair K of G with $d(K) = d(G)$.
2. $k \leftarrow d(G)$. Let $C_1, \ldots, C_k$ be the k components of $G - K$, and let ℓ_i ($i = 1, \ldots, k$) be the number of leaves of G contained in C_i.
3. Let $a_1, \ldots, a_k$ be k integers satisfying $\sum_{i=1}^{k} a_i = 2(k-1)$ and $\ell_i \leq a_i \leq b(C_i)$ for each $i = 1, \ldots, k$.
4. Construct a tree T with $V(T) = \{v_i \mid 1 \leq i \leq k\}$ satisfying $d_T(v_i) = a_i$ for each $v_i \in V(T)$.
5. $E^* \leftarrow \emptyset$. For each $(v_i, v_j) \in E(T)$, add an edge which connects a vertex in C_i with a vertex in C_j into E^* so that E^* finally satisfies that every leaf of G contains a vertex incident to some edge in E^* as well as b-feasibility of E^*. Output E^*.

Lemma 5: Algorithm 1 finds an optimum solution to 3VCA-DC in Case 1 and runs in $O(|V| + |E|)$ time. ■

Algorithm 2 /* for Case 2 */

1. Find an optimum solution F for 3VCA for G.
2. $E^* \leftarrow \emptyset$. For each $(u, v) \in F$, do the following: find two leaves $X(u), X(v) \in \mathcal{L}(G)$ containing u, v, respectively; select a vertex $u' \in X(u)$ with $b(u') \geq 1$ and a vertex $v' \in X(v)$ with $b(v') \geq 1$; and add an edge (u', v') into E^*. Output E^*.

Lemma 6: Algorithm 2 finds an optimum solution to 3VCA-DC in Case 2 and runs in $O(|V| + |E|)$ time. ■

4.2 Algorithm 3

In Case 3, we use the strategy of reducing the problem to Case 1 or Case 2 by adding several number of edges which can be a subset of an optimum solution. In Algorithm 3, at most two edges are added before the reduction. We omit the description of Algorithm 3 due to space limitation.

Lemma 7: Algorithm 3 finds an optimum solution to 3VCA-DC in Case 3 and runs in $O(|V| + |E|)$ time. ■

From Lemmas 5, 6 and 7, *Solve_3VCA-DC_aug2to3* finds an optimum solution and runs in $O(|V|+|E|)$ time. We obtain the following theorem.

Theorem 8: If a bi-connected graph G with degree constraints by b satisfies the existence condition for feasible solutions then finding an optimum solution F to 3VCA-DC for G and b can be done in $O(|V| + |E|)$ time and $|F| = \max\{d(G) - 1, \lceil l(G)/2 \rceil\}$. ■

5. Concluding Remarks

We have shown that checking the existence of a feasible solution and finding an optimum solution to 3VCA-DC for any bi-connected graph G can be done in $O(|V| + |E|)$ time. Devising a polynomial time algorithm for 3VCA-DC for not bi-connected graph is left for future research.

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