

# Edge Coloring Problem of Graph Theory Considering Interference on Network Coding

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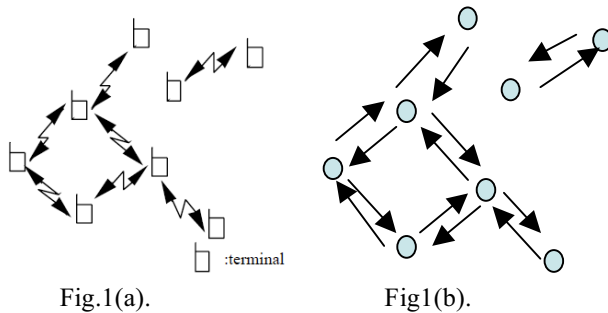
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**Abstract**—Multihop wireless networks consist of mobile terminals with personal communication devices. Each terminal can receive a message from a terminal and send it to the other terminal. If a terminal can not communicate the other terminal that sends data directly, some terminals relay the data. Network coding is a new architecture for wireless network and various applications using this architecture are expected. In this paper, we expand the previous coloring problem and propose a new coloring problem including the architecture.

*Keywords; Multihop wireless network, network coding, edge coloring, graph theory*

## 1. Introduction

Multihop wireless networks consist of mobile terminals with personal communication devices. Each terminal can receive a message from a terminal and send it to the other terminal. If a terminal can not communicate the other terminal that sends data directly, some terminals relay the data. In this network, each terminal communicates to the other terminals using a channel (Fig.1(a)). Since cochannel interference may occur, we can not assign channels randomly. We need to assign channels efficiently to achieve high spectral efficiency. In a previous paper[1], we formulated this problem using graph theoretical terms. In the formulation, we construct a graph in Fig.1(b) from Fig.1(a) and assign colors to edges.



Network coding[2] is a new architecture for wireless network and various applications using this architecture are expected. In this paper, we propose a new coloring problem

including the architecture and we propose a realistic cochannel interferences model.

## 2. Network coding and a new coloring

We consider the communication between vertices  $u$  and  $v$  in Fig.2.

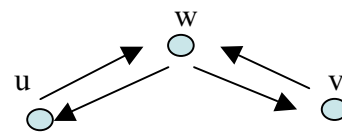
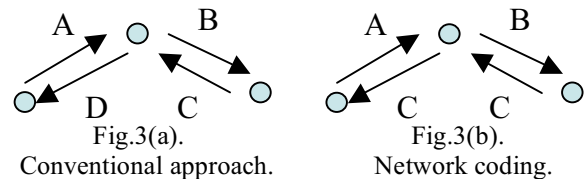


Fig.2. Communication between vertices  $u$  and  $v$ .

On conventional approaches, we need 4 steps (See Fig.3(a)). However, using concept of network coding, we can reduce it to 3 steps (Fig.3(b)). Because the packet from vertex  $w$  uses EX-OR of packets from  $u$  and  $v$  [3]. Each vertex can decode data from the received packet.



Here, we consider steps as colors. We formulate this problem as an edge coloring problem in graph theory.

[Definition 2]

Let  $D=(V,E)$  be a directed graph such that  $V$  is the vertex set and  $E$  is the edge set. A network coding coloring of  $D$  is an assignment of colors to edges of  $D$  satisfying the following conditions.

- 1) The edges that are incident to a vertex, receive different colors.
- 2) If edge  $e$  is incident to vertex  $u$ ,  $e'$  is incident from  $v$  and there is an edge  $(v,u)$  (See Fig.4),  $e$  and  $e'$  receive different colors.



Fig.4. Explanation of Definition 2.

Fig.5 is an example of the network coding coloring.

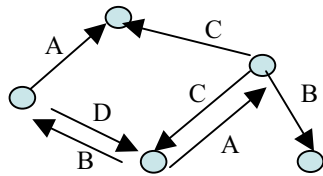


Fig.5. Network coding coloring.

In network coding coloring, same color edges from a vertex are allowed. Namely, this coloring corresponds to one-to-many communication. A coloring in case of one-to-one communication is discussed in [4]. And on undirected graphs, a coloring in case of one-to-one communication is called the strong edge coloring discussed in [5]. It is well-known that these coloring problems are NP-hard.

### 3. Results

#### 3.1 Previous studies and the extended results

A directed graph  $D$  is called symmetric, if whenever  $(u,v)$  is an edge of  $D$ , then so too is  $(v,u)$ . For example, the directed graph in Fig.2 is symmetric and the directed graph in Fig.5 is not symmetric.

In [6], we obtain the following theorem.

[Theorem 1] [6]

In symmetric directed tree  $T$  with maximum degree  $d_{max}$ , the optimal number of colors about network coding coloring is  $d_{max}/2 + 1$ .

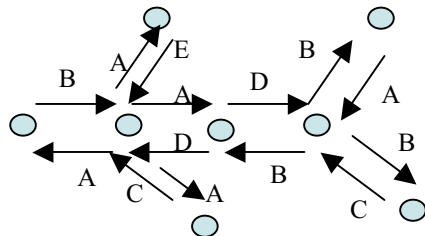


Fig.6. Graph G.

In the graph  $G$  of Fig.6,  $d_{max}/2 + 1 = 8/2 + 1 = 5$ . The coloring of Fig.5 is an optimal coloring.

In general directed trees, we obtain the following theorem.

[Theorem 2]

In a directed tree  $T$ , the optimal number of colors of  $T$  about network coding coloring is

$$\max \{ \text{din}(v) + \text{dou}(v) \mid v \text{ is in } V(T) \},$$

where  $\text{din}(v)$  represents the indegree of  $v$  and  $\text{dou}(v)$  is the following function,

$$\begin{aligned} \text{dou}(v) &= 1 \text{ if the outdegree of } v \text{ is greater than } 0, \\ \text{dou}(v) &= 0 \text{ if the outdegree of } v \text{ is } 0. \end{aligned}$$

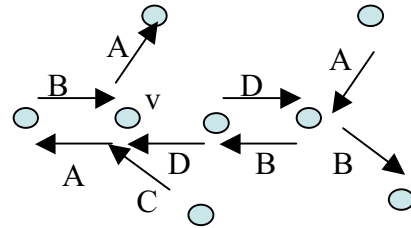


Fig.7. Graph  $G'$ .

(The outline of the proof)

In the graph  $G'$  of Fig.7,  $\text{din}(v) + \text{dou}(v) = 3 + 1 = 4$  that is the maximum value. On a coloring of  $G'$ , we obviously need 4 colors. Therefore, the coloring of  $G'$  of Fig.7 is an optimal coloring.

Please look at  $G$  of Fig.6 and  $G'$  of Fig.7.  $G'$  is a subgraph of  $G$ . The color assigned to each edge of  $G'$  is same to the color assigned to the edge of  $G$ . Namely, the assignment of  $G$  can be used to  $G'$ . ///

Here, we consider a vertex coloring related to network coding coloring (A similar discussion is in [7]).

Let  $V' = \{v \mid \text{dou}(v) > 0\}$  and we assign color to each vertex in  $V'$ . The condition is the following.

1) we assign different colors to  $u$  and  $v$ , if  $u$  is adjacent from/to  $v$  (Fig.8(a)).

2) we assign different colors to  $u$  and  $v$ , if there is a vertex  $w$  that is adjacent from  $u$  and  $v$  (Fig.8(b)).

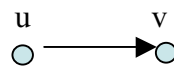


Fig.8(a).

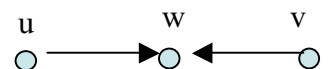


Fig.8(b).

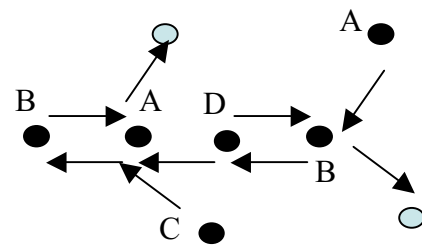


Fig.9. A vertex coloring.

In Fig.9, black vertices are in  $V'$  and the vertex coloring satisfies the above condition. Here, we set an edge coloring as follows. For each directed edge  $(u,v)$ , we assign a color of vertex  $u$  to  $(u,v)$ . Then, the edge coloring of Fig.9 is same to the edge coloring of Fig.7.

So, we can consider this coloring problem as a vertex coloring. However, this vertex coloring modeling can not correspond to a model having a limitation of the number of simultaneous communication terminals as Bluetooth.

Therefore, we consider this coloring problem as edge colorings.

### 3.2 More realistic cochannel interferences model

It was the assumption that there was not the interference between two vertices if there was not an edge between the vertices till now. Hereafter, we consider the more realistic cochannel interferences model. For example, in Fig.10, we assume that the communication from  $u$  to  $v$  influences the communication of  $w$ . In other words, we can not assign a same color to  $(u,v)$  and  $(x,w)$ .

Under the above assumption (we call this assumption Assumption A), we obtain the optimal number of colors of a tree.

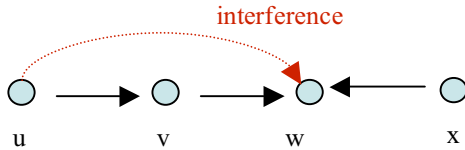


Fig.10. Explanation of interference.

[Theorem 3]

In symmetric directed tree  $T$ , the optimal number of colors about network coding coloring under the Assumption A is  $\max \{ \text{din}(u) + \text{din}(v) \mid (u,v) \text{ is in } E(T) \}$ .

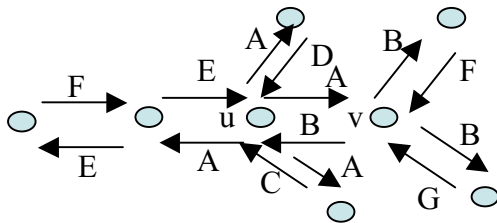


Fig.11. Network coding coloring.

In the graph of Fig.11,  $\text{din}(u) + \text{din}(v) = 4+3 = 7$  that is the maximum value. From Theorem3, the coloring of Fig.11 is an optimal coloring.

(The outline of the proof)

Let  $u$  and  $v$  be vertices of  $T$  such that  $(u,v)$  is an edge and  $\text{din}(u) + \text{din}(v)$  is the maximum value. Let  $a=\text{din}(v)$  and  $b=\text{din}(u)$ . First we assign colors to edges incident from/to  $u$  or  $v$ . we assign colors  $1,2,\dots,a$  to  $(u,v),(u_2,v),\dots,(u_a,v)$  and we assign colors  $a+1,\dots,a+b$  to  $(v,u),\dots,(v_{a+b},u)$ . And we assign color 1 to edges incident from vertex  $u$  and color  $a+1$  to edges incident from vertex  $v$  (Fig.12).

Next, we assign colors  $2,\dots,a+b$  to edges incident from  $u_2,\dots,v_{a+b}$ , respectively. Here we consider assigning colors to edges incident to  $u_i$ , excepting  $(v,u)$ . The number of edges incident to  $u_i$  is not greater than  $b-1$  because  $\text{din}(u) + \text{din}(v)$  is the maximum value. Therefore, we assign colors  $a+2,\dots,a+b$  to edges incident to  $u_i$ . Since we can expand this coloring to other edges, the total number of colors is  $a+b=\text{din}(u) + \text{din}(v)$ . ///

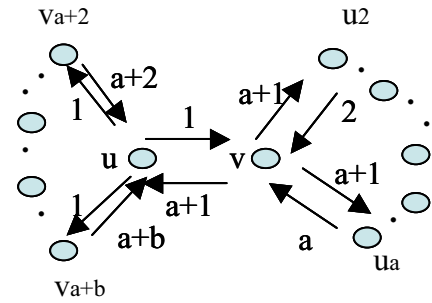


Fig.12. Explanation of the proof of Theorem 3.

In general directed trees, we obtain the following theorem.

[Theorem 4]

In a directed tree  $T$ , the optimal number of colors of  $T$  about network coding coloring under the Assumption A is the maximum value of the sum of the following values for each vertex  $v$ .

- (a)  $\text{din}(v)+\text{dou}(v)$
- (b)  $\max \{ \text{dou}(u) \mid (v,u) \text{ is in } E(T) \}$
- (c)  $\max \{ \max \{ \text{din}(u) - 1 \mid (u,v) \text{ is in } E(T) \}, \max \{ \text{din}(u) \mid (v,u) \text{ is in } E(T) \text{ and } (u,v) \text{ is not} \} \}$

In the graph of Fig.13, the coloring is an optimal coloring. The colors corresponding to (a), (b) and (c) are as follows.

- (a)  $\text{din}(v)+\text{dou}(v) : A,B,C$
- (b)  $\max \{ \text{dou}(u) \mid (v,u) \text{ is in } E(T) \} : D$
- (c)  $\max \{ \max \{ \text{din}(u) - 1 \mid (u,v) \text{ is in } E(T) \}, \max \{ \text{din}(u) \mid (v,u) \text{ is in } E(T) \text{ and } (u,v) \text{ is not} \} \} : E,F$

(The outline of the proof)

The details of the assignment of Fig.13 are as follows.

- (a) we assign color A to the edges incident from  $v$  and colors B,C to the edges incident to  $v$ .
- (b) If there is an edge  $(v,u)$  and edge  $e$  incident from  $u$ , we assign color D to  $e$ .
- (c) If  $(u,v)$  is in  $E(T)$ , The number of colors assigned to edges incident to  $u$  is  $\text{din}(u)-1$  since color A is already assigned to  $(v,u)$ . If  $(u,v)$  is not in  $E(T)$ , The number of colors assigned to edges incident to  $u$  is  $\text{din}(u)$ .

This coloring is under the Assumption A and we can expand this coloring to other edges. ///

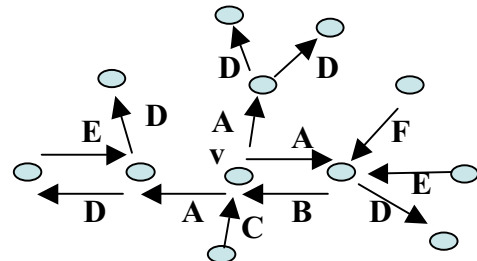


Fig.13. Network coding coloring.

In practice, although a terminal  $v$  can communicate with the other terminal  $u$ ,  $v$  may not communicate with  $u$ . Therefore,

we divide the edge set into communication edges and interference edges. We assign colors to only communication edges. Moreover, it is hard to think about the following situation.

Terminal  $v$  receives the interference from terminal  $u$ .

However,  $u$  does not receive the interference from  $v$ .

So we assume that graphs are symmetric, hereafter. According to this assumption, we define the more general coloring problem in this paper.

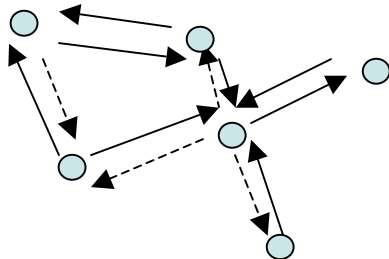


Fig.14. New model G of network coding coloring.

In Fig.14, solid lines represent communication edges and dotted lines represent interference edges. A network coding coloring is in Fig.15.

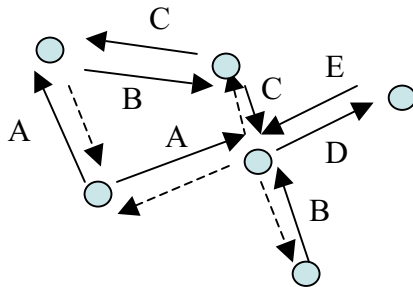


Fig.15. Network coding coloring of G.

We can express Assumption A by adding interference edges as follows.

If  $(u,v)$  and  $(v,w)$  are edge of the graph, then we draw  $(u,w)$  as an interference edge (See Fig.10). For example, in case of Fig.15, Fig.16 is the graph adding interference edges. The bold dotted lines represent added interference edges.

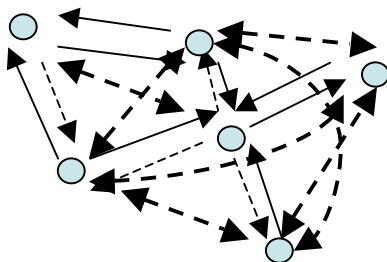


Fig.16. Explanation of Assumption A.

In this modeling, we obtain results similar to Theorem 1-4.

## 4. Conclusion

In this paper, we consider a new edge coloring in directed graphs related to network coding. We introduce the relation between the previous studies and the edge coloring. And we extend the previous results. Then, we propose more realistic cochannel interferences model and show some result about this model.

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