

Modeling of Quantum Computer by using Quantum Petri Net

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Abstract: Petri Net that is an effective tool for concurrent systems, has been extended to Quantum Petri Net. The conventional modeling method of quantum computer using Quantum Petri Net, have several drawbacks. This paper defines new model and examines conditions to solve these drawbacks, and proposes new model from modeling quantum Turing machine, and shows that the model can simulate more general calculation processes than which the conventional model.

1. Introduction

Quantum computer (QC)[1], [2], [3] has been expected to run faster than classical ones since the algorithms for prime factorization and discrete logarithm were found. In research of quantum computer, computation models, for example, finite automata (FA) and Turing machine (TM), have been extended to quantum ones so that their state can be superposed, and quantum computation power is being studied by using these extended model.

On the other hand, Petri Net (PN)[4], [5], that is an effective tool for modeling and analysis of concurrent systems, has been extended to Quantum Petri Net (QPN)[6], [7]. PN is superclass of FA, and superior to TM in solvability of analysis problems[4]. QPN is expected to be similarly superior in modeling power and solvability of analysis problems, and needs to be studied on both about modeling and analysis.

In Ref.[6], [7], Tsuji et al. have proposed a model of quantum bit (qubit) [8], which is corresponding to bit in classical computation, and tried to model quantum Turing machine (QTM)[2], [3] by using the qubit model. However those definition and modeling method have several drawbacks with modeling quantum computation in more general calculation process.

In this paper, first, we define new model and examine conditions for modeling in order to solve these drawbacks. Next, we propose a new model from examination of modeling QTM as one of QC, and show that this model can simulate more general calculation processes than the conventional model.

2. Preliminaries

First, we define QTM as one of QC. And in order to simplify modeling, add several conditions to QTM.

Definition 1 Quantum Turing Machine[2], [3]

Quantum Turing Machine (QTM) is defined by a triplet $M = (\Sigma, Q, \delta)$, where Σ is a finite alphabet with an identified blank symbol $\#$, Q is a finite set of states with an identified initial state q_0 and final state $q_f \neq q_0$, and δ is quantum transition function which is a mapping from $Q \times \Sigma \times \Sigma \times Q \times \{L, R\}$ to the set of complex number \mathbf{C} . \square

In the same way as TM, QTM has finite state, tape head, and infinite tape (with integral index). QTM is physically or mathematically represented as follows,

$$|\psi(\tau t)\rangle = U_M^t |q_0; 1; S_0\rangle,$$

where S_0 is the initial tape state. Then, $|\psi(\tau t)\rangle = \sum_{x \in \{0,1\}^n} c_x |x\rangle$, $c_x \in \mathbf{C}$, and sum of $|c_x|^2$ equals to 1.

In this paper, we assume that tape is finite, and call QTM which has n -qubit tape n -QTM. The reason for assuming finite tape is to consider that QTM is a modeling object and the following that one of QC. And in order to simplify modeling, we add conditions to QTM.

1. The direction of tape head is right only.
2. If the tape head reaches right tape end, it goes to left tape end.
3. The tape symbol is either 0 or 1, and the tape state is represented by a binary number.
4. The tape state transition take place between their tape states, where the Hamming distance between the value of their tape states equals to 1.

So we consider that the number of qubits which n -state tape has is $\log n$. Moreover, we give the activity of QTM by using calculation tree.

Next, we define QPN. A QPN is defined using an Extended Petri Net (EPN)[6], [7].

Definition 2 Quantum Petri Net[6], [7]

Extended Petri Net (EPN) is defined by five tuple $EPN = (P, T, F, W, M_0)$, where P is a finite set of places, T is a finite set of transitions, $F \subseteq (P \times T) \cup (T \times P)$ is a set of arc, W is weight function which is a mapping from F to \mathbf{C} , and M_0 is the initial marking and mapping from P to \mathbf{C} .

[Token] The symbol \bullet^w denotes a token with weight w which is some complex number. When $w = v \cdot y$, \bullet^w is equivalent to $v \cdot \bullet^y$, where v is an integer.

[Transition] The symbol \square denotes the choice firing transition. Choice firing transition is that we can decide whether the transition fires or not. The symbol \blacksquare denotes the free firing transition. Free firing transition fires when it is enabled.

[Firing rule] By a firing of enabled transition t at a marking M , the token is moved and the resulting new marking $M' \geq 0$ is defined as follows, where $a = \min |M(p)|$, $p \in \bullet t$, and $a \cdot w$ implies that the weight u of each moved token change to $u \cdot w$.

$$M'(p) = M(p) + a \cdot W(t, p) : p \in t \bullet \wedge p \notin \bullet t,$$

$$M'(p) = M(p) - a \cdot W(p, t) : p \in \bullet t \wedge p \notin t \bullet,$$

$$M'(p) = M(p) : \text{otherwise.}$$

[Quantum Petri Net] An EPN N is called a Quantum Petri Net (QPN) when each transition in N can fire only at integer time instances, and enabled transitions are not in conflict with one another. \square

3. Conventional modeling method by QPN

Let us show conventional modeling method[6], [7]. First, we show a modeling of qubit as the basis of the model. Qubit is represented as follows

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle, |c_0|^2 + |c_1|^2 = 1$$

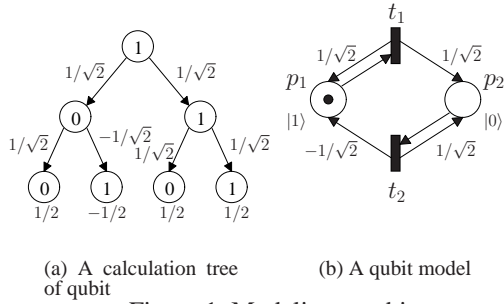


Figure 1. Modeling a qubit

Table 1. Token's weight and Probability of Fig.1

Times	0	τ	2τ	3τ	4τ	5τ	6τ	7τ	8τ
$M(p_1)$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1
$M(p_2)$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0
$P(p_1)$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1
$P(p_2)$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0
$P(p)$	1	1	1	1	1	1	1	1	1

This is a linear combination of $|0\rangle$ and $|1\rangle$ which consist the orthonormal basis of \mathbf{C}^2 . This basis $|0\rangle$ and $|1\rangle$ are corresponding to 0 and 1 of bit respectively, and c_1, c_2 are called probability amplitudes or amplitudes in short. Then, $|c_i|^2, (i \in \{0, 1\})$ is probability of that qubit state $|\psi\rangle$ is $|i\rangle$. In modeling qubit, vector $|0\rangle, |1\rangle$ and its probability amplitude is modeled by places and weights of token, and change of probability amplitude is modeled by a transition and its weight of output arc.

Example 1 Qubit model[6], [7]

We show a qubit model in Fig.1(b) when change of amplitude is as shown Fig.1(a). And, show in Tab.1 each token's weight and probability at each integer time instance in a simulation of this model, where τ is a unit and $P(p)$ is probability of that $|\psi\rangle$ being $|0\rangle$ or $|1\rangle$.

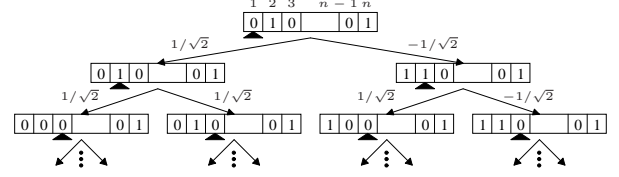
Next, we show conventional modeling method of QTM using the qubit model. In Fig.2(a), we show a calculation tree of n -QTM. Each node implies a tape state S_i , each arc from S_i to S_j implies change of state from S_i to S_j , and each label on the arc implies change of amplitude. In modeling of QTM, places modeling tape head of QTM control the qubit model, and the amplitude of each tape state is the sum-product of each token's weight.

Example 2 Modeling of a QTM

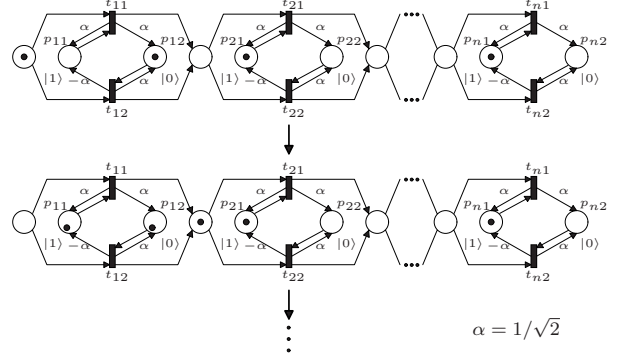
We show in Fig.2(b) the QTM model which is shown in Fig.2(a). For example, the amplitude of the state $|0001\dots 01\rangle$ on Level 2 is $1/\sqrt{2} \cdot 1/\sqrt{2} = 1/2$ because $M(p_{12}) = M(p_{22}) = 1/\sqrt{2}$. The probability of the state $|0001\dots 01\rangle$ is $(1/2)^2 = 1/4$ and the probability of this level is $1/4 + 1/4 + 1/4 = 1$.

4. Problems of conventional method

We show problems for modeling QTM in more general calculation processes than which conventional model can simulate. **[Drawback 1]** We can not model quantum entanglement involving plural quantum objects like qubits. For example, the QTM state of Level 4 in the Fig.3(a) is only two state of $|00\rangle$ and $|11\rangle$ since entanglement of qubit exists. The state can not be represented as the sum-product of token's weight as Example 2.

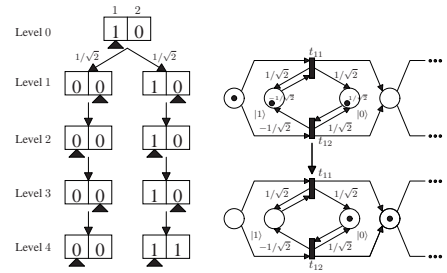


(a) A calculation tree of QTM



(b) Modeling a QTM by qubit model

Figure 2. Conventional modeling of a QTM



(a) A calculation of a 2-QTM (b) A conflict and real activity

Figure 3. Examples of calculations that can not be modeled using conventional method

[Drawback 2] Existence of conflict between transitions (See t_{11} and t_{12} in the Fig.3) controlled by a place representing the tape head, that should fire simultaneously to calculate probability amplitudes of states correctly and/or to achieve parallelism inherent to quantum computation.

[Drawback 3] Choice firing transitions can not express simultaneity of firing, and it is difficult for free firing transitions to express choice.

5. Examination of modeling QTM by new model

When we consider modeling calculation process like as shown in Fig.3, the preliminary idea is that we extend the definition and net structure of model. However, the size of model based on a qubit model grows more than exponentially to the number of qubits because of quantum entanglement. Hence, in this paper, we consider a model based on a change of state expressed by qubits.

5.1 A model based on change of state expressed by qubits

First, we examine and give conditions of modeling object.

[Condition 1] The number of tape state is n , and modeling object is $\log n$ -QTM which has $\log n$ -qubit tape. It is because, the size of places must be polynomial of the number of tape state, when we consider a tape state of QTM modeled by a place.

[Condition 2] The value of probability amplitude of state change is limited to a member of a finite set of complex numbers.

[Condition 3] There is only one resource (i.e. the sum of square of probability amplitude equals to 1). It is needed to avoid of drawback of plural quantum objects (Drawback 1). Furthermore, we assume that the object of simulation is an activity of QTM which is given by a calculation tree.

Next, we examine and give definition and restriction of the new model. Because of drawback of synchronization (Drawback 2), we restrict the model to class as follows.

Definition 3 Quantum state machine

Quantum state machine (QSM) is an EPN N which satisfies the following.

$$\forall t \in T; |\bullet t| = 1 \wedge \sum_{p \in \bullet t} |w(p, t)|^2 = \sum_{p \in t \bullet} |w(t, p)|^2 = 1$$

□

Moreover, in order to avoid the drawback when using choice firing transition (Drawback 3), we define new firing rule as follows.

Definition 4 Firing rule of choice firing transition

1. Each enabled transition can fire at integer time instances. This time is called enable time.
2. At an enable time, transitions which are not in conflict with one another can fire simultaneously.
3. When enabled transitions fire, the change of marking is according to QPN's rule.

□

Lastly, for simulation of QSM, we define firing rule about choice and conflict as follows, because the conditions of QPN includes the condition about time and conflict.

Definition 5 Firing rule of QSM

Firing rule of QSM is defined as, in addition to the Definition 4, at least one transition is chosen from a set of output transition of a place p where $M(p) \neq 0$, and each transition must fire at the time when it is enabled.

□

Definition 6 Simultaneous choosable transition set

Simultaneous choosable transition set (SCT set), is a subset of T , whose transitions are not in conflict with one another. When st is an SCT set, it satisfies the following.

$$\forall t, t' \in st; \bullet t \cap \bullet t' = \emptyset$$

□

Definition 7 Choice of transition in QSM

A set of SCT set ST is given to QSM. ST satisfies the following.

$$\forall st \in ST; \bullet st = P$$

At enable time, a simultaneous choosable set $st \in ST$ is chosen, and all enabled transitions in st fire.

□

Under the above conditions, restrictions, and definitions, we examine modeling as follows.

Modeling 1 $\log n$ -QTM

We model $\log n$ -QTM, where S is a set of tape state, and $|S| = n$. The modeling conditions is as follows.

1. Each amplitude of tape state is modeled by a weight of the corresponding token.
2. The amplitude of state transition change from tape state S_i to S_j is modeled by a weight of an arc $w(t, p_j)$, where S_i is modeled by place p_i , and $t \in p_i \bullet \cap \bullet p_j$.
3. The finite state and tape head of QTM is modeled by choice of transitions.
4. The probability of tape state is the square of its amplitude.
5. The incidence matrix which is led by st and $st \bullet$ is an unitary matrix, where st is a SCT set chosen at some time.

We show the modeling algorithm as follows.

[Algorithm 1] Modeling of $\log n$ -QTM

Define the variables as $x \in \{0, 1\}^{\log n}$, $y \in \{0, 1\}^{\log n}$, $z \in \{0, 1\}^{\log n-1}$, $k = \{1, 2, \dots, \log n\}$, $U = \{U_1, U_2, \dots, U_m\}$ which is the set of unitary matrix, where m and n are integers, and $u_{a,ij}$ is the (i, j) -component of the unitary matrix U_a .

1. $k = 1$.
2. $z = 0$.
3. Let $|x\rangle$ be the state resulted by inserting $|0\rangle$ in $|z\rangle$ as the k -th qubit.
4. Let $|y\rangle$ be the state resulted by inserting $|1\rangle$ in $|z\rangle$ as the k -th qubit.
5. $a = 1$.
6. Do bijection $\{|x\rangle\} \mapsto p$, $\{|x\rangle, k, U_a\} \mapsto t$, $w(p, t) = 1$, $w(t, p) = u_{a,00}$. And do bijection $\{|y\rangle\} \mapsto p$, $w(t, p) = u_{a,10}$.
7. If $a < m$, $a = a + 1$, and go to 6. Else, next.
8. If $z < 2^{\log(n-1)}$, $z = z + 1$, and go to 3. Else, next.
9. If $k < \log n$, $k = k + 1$, and go to 2. Else, end.

n is the number of tape state, and m is the number of unitary matrices (2×2). $|x\rangle$ and $|y\rangle$, k , and U are the variables corresponding to the tape state, position of tape head, unitary matrix, respectively. This algorithm converges and the complexity is polynomial of m, n (namely $mn \log n$).

Example 3 Modeling of a 2-QTM

We use algorithm for modeling 2-QTM, and simulate the activity of 2-QTM as shown in Fig.3(a). The setting of algorithm is that $n = 4$, $m = 3$, and $U = \{U_1, U_2, U_3\}$ is given as follows

$$U_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, U_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, U_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Moreover, each variable is corresponding to the graph node as follows

$$\{|x\rangle\} \leftrightarrow p_{(x_d+1)}, \{|x\rangle, k, U_a\} \leftrightarrow t_{(x_d+1),k,a},$$

where x_d is the decimal expression of binary number x . Under this setting, the graph generated by this algorithm is shown in Fig.4, and simulation result is shown in Tab.2 when we choose SCT set st_1, st_2, st_3, st_4 in turn, where $st_1 = \{t_{111}, t_{211}, t_{311}, t_{411}\}$, $st_2 = \{t_{123}, t_{223}, t_{323}, t_{423}\}$, $st_3 = \{t_{113}, t_{213}, t_{313}, t_{413}\}$, $st_4 = \{t_{123}, t_{223}, t_{322}, t_{422}\}$. For example, at time τ and initial marking $M_0 = [0, 0, 1, 0]^T$, t_{311} in st_1 is enabled. Then, t_{311} fires, and M_0 changes to $M_1 = [1/\sqrt{2}, 0, 1/\sqrt{2}, 0]^T$.

□

We have checked that this modeling is correct by simulation of the qubit and find this model can simulate a process which conventional model can not simulate. For example,

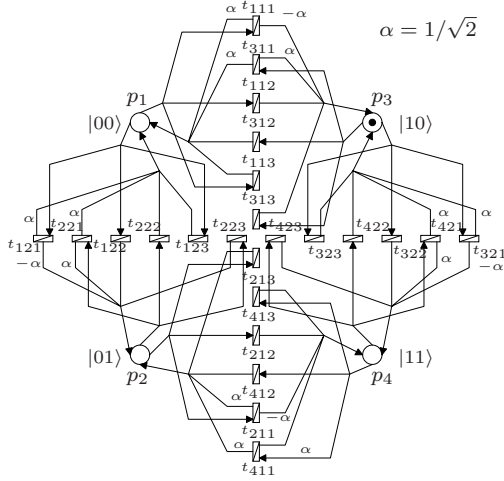


Figure 4. Modeling of a 2-QTM (Initial marking M_0)

Table 2. Token's weight and probability of Fig.4

Times	0	τ	2τ	3τ	4τ
$M(p_1)$	0	$1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$
$M(p_2)$	0	0	0	0	0
$M(p_3)$	1	$1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$	0
$M(p_4)$	0	0	0	0	$1/\sqrt{2}$
$P(p_1)$	1	1/2	1/2	1/2	1/2
$P(p_2)$	0	0	0	0	0
$P(p_3)$	0	1/2	1/2	1/2	0
$P(p_4)$	0	0	0	0	1/2
$P(\cup_{p \in P} p)$	1	1	1	1	1

this model can simulate the process shown in Fig.3(a). Moreover, the time complexity of algorithm and the size of net are polynomial of the number of tape state. However, we need to restrict SCT set when we choose transitions to fire, because the number of its elements is exponential to the number of tape states.

5.2 Corresponding to mathematical model and problem of new model

When we only focus on the tape state change in Fig.3(a), we consider that the following unitary transformations are applied in turn.

$$\begin{aligned} A_1 &= U_1 \otimes U_3, \\ A_2 = A_3 &= U_3 \otimes U_3, \\ A_4 &= |0\rangle\langle 0| \otimes U_3 + |1\rangle\langle 1| \otimes U_2 \end{aligned}$$

The transposed matrices of A_1 to A_4 equal to the forward incidence matrices led by SCT set which is used in Example 3. Moreover, A_1 to A_4 can be given by quantum gates[9].

However, when we consider correspondence to mathematical model, there is a problem. Now, finite state, tape head, and tape state of QTM is a superposition as follows.

$$|\psi(\tau t)\rangle = \dots + c_1|q_1, k, S_1\rangle + c_2|q_2, k, S_1\rangle + \dots$$

Then, on our proposed model, marking of p_1 , which is corresponding to S_1 , is $M(p_1) = c_1 + c_2$, because these tape states are S_1 . In practice, however, these amplitudes do not interfere, because these finite states and positions of tape head are

different. This model has such a problem when chooser of choice firing transition is quantum resource.

5.3 Analysis and necessity to re-examine definition

When we consider the analysis of quantum systems, the analyzed properties of the systems may be the unitarity of its state change, the change and interference of amplitude of its state, and as simulation analysis, quantum error.

For example, if we try to analyze the unitarity of state change, we may use graph search of our proposed model, since each state and its change is corresponding to a place and token moving of our model, respectively. Petri Net model is not useful for this analysis. Hence, we consider that let the system state be corresponding to the combination of the number of token whose each place has like as conventional Petri Net, and we analyze this model using analysis tree, and search of graph and marking. For this analysis, we need to consider dividing weight of token and the number of token. However, if we let the system state be corresponding to the combination of the number of token, the interference of amplitude takes place between the same combinations since it take place between the same states of system. In this case, we let conventional marking be weighted. If we try to analyze the change and interference of amplitude, we can consider using state equation, however, there do not exist it now.

6. Conclusion

In this paper, first, we show problems of conventional modeling method, and an example of calculation process that we can not model by this method. Next, we propose a new model in order to solve the problems, and show that the model can simulate more general calculation processes which is represented by calculation tree than conventional model can simulate.

As a result of examination, we find the problem of new model that the handling rules of quantum mechanics involving resources and chooser of choice firing transition. Furthermore, new definition of model need to be re-examined about the number of token and a weight of token for analysis. So as future studies, we need to re-examine the definition and examine about concrete analysis.

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