Conductivity Image Reconstruction Based on Singular Value Decomposition Method in Electrical Impedance Tomography

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Abstract: In this paper, an inverse method based on singular value decomposition is proposed to solve the inverse problem with the generalized Tikhonov regularization to determine the internal conductivity distribution in electrical impedance tomography. Numerical simulations have been carried out to evaluate the performance of the proposed method.

Keywords—**Electrical impedance tomography, Singular value decomposition, Tikhonov regularization, Image reconstruction**

1. Introduction

As one of tomographic imaging modalities, electrical impedance tomography (EIT) has been used to monitor two-phase flow processes [1], because of its low cost and high temporal resolution characteristics for monitoring transient processes. EIT is a noninvasive imaging modality for determining the electrical properties inside the domain of interest. Currents are injected and the resultant boundary voltages are measured through the electrodes attached on the periphery of the domain. The internal conductivity distribution is reconstructed based on these current and voltage data.

EIT consists of forward and inverse problems. The forward problem is usually solved based on numerical techniques such as finite element method and boundary voltages are computed based on injected currents and given conductivity distribution. The inverse problem is to determine unknown conductivity distribution based on injected currents and measured boundary voltages [2], [3]. The relationship between the internal conductivity distribution and boundary voltages is nonlinear. Therefore, to obtain good resolution of reconstructed images, most inverse solvers employ iterative algorithms such as iterative Gauss–Newton (iGN) method, which are obtained by linearizing the nonlinear inverse problem. The iGN method estimates the conductivity distribution by computing Jacobian matrix on every iteration. However, in the case of monitoring fast transient processes, iterative methods may fail to reconstruct the actual characteristics of flows because the Jacobian matrix is computed on every iteration.

Therefore, to monitor fast transient processes in a binary mixture flows online, usually one-step algorithms are preferred to iterative algorithms due to less computational time. These one-step methods are based on the assumption that the varying conductivity distribution differs only slightly from homogeneous distribution. With this assumption, the inverse problem is linearized and the Jacobian matrix is computed beforehand. After that, the unknown conductivity distribution is reconstructed with new experimental data, for example back-projection [4], NOSER [5], linearization method [6], one-step Gauss–Newton (oGN) method and so on.

Usually, in the one-step inverse methods, the standard Tikhonov regularization is used to make the inverse problem well-posed. Moreover, to improve the resolution of the reconstructed images, different regularization operators can be applied instead of the identity matrix as the regularization matrix. However, for any regularization parameter, the inverse matrix should be computed and it takes some time to determine the conductivity distribution in the binary mixture flow domain.

To overcome this drawback, singular value decomposition (SVD) method can be used. In the case of the standard Tikhonov regularization, the SVD method can be applied directly. However, when the inverse problem has the generalized Tikhonov regularization, the generalized SVD (GSVD) method is applicable instead of the SVD method [7].

In this paper, to monitor the internal conductivity distribution online, an inverse method based on the SVD method is proposed to solve the inverse problem with the generalized Tikhonov regularization. To reduce the online computational time, the generalized regularization matrix is taken out from the inverse computation term and the SVD method is applied. Numerical simulations have been carried out to validate the performance of the proposed method and their results are compared with the conventional SVD method.

2. Image Reconstruction Method

2.1 Linearization

In EIT, measured and calculated boundary voltage data have the following relation

$$V = U(\sigma) + w \tag{1}$$

where $V \in \mathbb{R}^{M\times 1}$ is measured voltage vector and $U(\sigma) \in \mathbb{R}^{M\times 1}$ is voltage vector calculated by using finite element method, σ is the conductivity distribution in a given domain, w is the error and M is the number of measurements.

Linearizing (1) with the first order Taylor polynomial at an initial conductivity σ_0 and omitting higher order terms including the error, the following equation is obtained

$$V = U(\sigma_0) + \mathbf{J}(\sigma_0)(\sigma - \sigma_0) \tag{2}$$

where $\mathbf{J}(\sigma_0) \equiv \partial U(\sigma_0)/\partial\sigma_0 \in \mathbb{R}^{M\times N}$ is the Jacobian matrix and N is the number of elements in a finite element mesh. The initial guess σ_0 can be obtained from the best constant conductivity approximation that is the reciprocal of the resistivity approximation [5]. Rewriting (2), the linear equation is obtained

$$\delta V = \mathbf{J}_0 \delta\sigma \tag{3}$$

where $\delta V \equiv V - U_0$, $U_0 \equiv U(\sigma_0)$, $\mathbf{J_0} \equiv \mathbf{J}(\sigma_\mathbf{0})$ and $\delta\sigma \equiv \sigma - \sigma_0$.

2.2 Conventional method

In the linear equation (3), there is no pseudo-inverse because of ill-posedness. Therefore, to solve this equation, the regularization should be employed to make it well-posed.

Usually, the standard Tikhonov regularization is used in the inverse EIT problem. Therefore, with the aid of the regularization, we have the so-called oGN method as

$$\hat{\sigma} = \sigma_0 + \left(\mathbf{J}_0^T\mathbf{J}_0 + \alpha^2\mathbf{I}\right)^{-1}\mathbf{J}_0^T\delta V \tag{4}$$

where $\hat{\sigma} \in \mathbb{R}^{N\times 1}$ is updated conductivity distribution, α is the regularization parameter and $\mathbf{I} \in \mathbb{R}^{N\times N}$ is the identity matrix.

For any α in (4), the inverse matrix should be computed and it takes more time to determine the internal conductivity distribution in the binary mixture flow domain.

To resolve the drawback to it, the SVD method can be used as another expression of (4).

$$[\mathbf{u}, \mathbf{s}, \mathbf{v}] = \mathrm{svd}(\mathbf{J}_0) \quad \Leftrightarrow \quad \mathbf{J}_0 = \mathbf{usv}^T \tag{5}$$

where $\mathbf{s}$ is a non-negative diagonal matrix with singular values of $\mathbf{J}_0$, $\mathbf{u}$ and $\mathbf{v}$ are unitary matrices where $\mathbf{u}^T\mathbf{u} = \mathbf{v}^T\mathbf{v} = \mathbf{I}$. Therefore, with the aid of the SVD method, we have

$$\hat{\sigma} = \sigma_0 + \sum_{i=1}^{N_s} \frac{s_i}{s_i^2 + \alpha^2} u_i^T \delta V v_i \tag{6}$$

where s_i are diagonal elements of $\mathbf{s}$, u_i and v_i are column vectors of $\mathbf{u}$ and $\mathbf{v}$, respectively, N_s is the number of diagonal elements of $\mathbf{s}$.

2.3 Proposed method

To improve the resolution of the reconstructed images, a different regularization matrix can be applied instead of the identity matrix in (4). When applying the generalized Tikhonov regularization in (3), we have

$$\hat{\sigma} = \sigma_0 + \left(\mathbf{J}_0^T\mathbf{J}_0 + \alpha^2\mathbf{P}\right)^{-1}\mathbf{J}_0^T\delta V \tag{7}$$

where $\mathbf{P} = \mathbf{R}^T\mathbf{R}$, $\mathbf{R} \in \mathbb{R}^{N\times N}$ is the regularization matrix that is a first-order discrete Gaussian smoothing operator [8].

For the generalized Tikhonov regularization, the GSVD method can be applied [7]. However, in this paper, we solve the inverse solver (7) using the SVD method.

Rewriting the matrix inversion term in (7), we have

$$\begin{aligned}
\left(\mathbf{J}_0^T\mathbf{J}_0 + \alpha^2\mathbf{P}\right)^{-1} &= \left(\mathbf{J}_0^T\mathbf{J}_0 + \alpha^2\mathbf{P}^{\frac{1}{2}}\mathbf{P}^{\frac{1}{2}}\right)^{-1} \\
&= \left[\mathbf{P}^{\frac{1}{2}}\left(\mathbf{P}^{-\frac{1}{2}}\mathbf{J}_0^T\mathbf{J}_0\mathbf{P}^{-\frac{1}{2}} + \alpha^2\mathbf{I}\right)\mathbf{P}^{\frac{1}{2}}\right]^{-1} \\
&= \tilde{\mathbf{P}}\left(\tilde{\mathbf{P}}\mathbf{J}_0^T\mathbf{J}_0\tilde{\mathbf{P}} + \alpha^2\mathbf{I}\right)^{-1}\tilde{\mathbf{P}} \\
&= \tilde{\mathbf{P}}\left(\tilde{\mathbf{J}}^T\tilde{\mathbf{J}} + \alpha^2\mathbf{I}\right)^{-1}\tilde{\mathbf{P}}
\end{aligned} \tag{8}$$

where $\tilde{\mathbf{P}} \equiv \mathbf{P}^{-\frac{1}{2}} \in \mathbb{R}^{N\times N}$, $\tilde{\mathbf{J}} \equiv \mathbf{J}_0\tilde{\mathbf{P}} \in \mathbb{R}^{M\times N}$ and $\mathbf{I} \in \mathbb{R}^{N\times N}$ is the identity matrix.

Therefore, the inverse solver (7) can be rewritten as

$$\hat{\sigma} = \sigma_0 + \tilde{\mathbf{P}}\left(\tilde{\mathbf{J}}^T\tilde{\mathbf{J}} + \alpha^2\mathbf{I}\right)^{-1}\tilde{\mathbf{J}}^T\delta V \tag{9}$$

To avoid computing the inverse matrix for any α in (9), the SVD method can be applied.

$$[\tilde{\mathbf{u}}, \tilde{\mathbf{s}}, \tilde{\mathbf{v}}] = \mathrm{svd}(\tilde{\mathbf{J}}) \quad \Leftrightarrow \quad \tilde{\mathbf{J}} = \tilde{\mathbf{u}}\tilde{\mathbf{s}}\tilde{\mathbf{v}}^T \tag{10}$$

where $\tilde{\mathbf{s}}$ is a non-negative diagonal matrix with singular values of $\tilde{\mathbf{J}}$, $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ are unitary matrices.

Therefore, for the generalized Tikhonov regularization, the proposed method based on the SVD method can be obtained as

$$\hat{\sigma} = \sigma_0 + \tilde{\mathbf{P}}\tilde{\sigma}, \quad \tilde{\sigma} = \sum_{i=1}^{N_s} \frac{\tilde{s}_i}{\tilde{s}_i^2 + \alpha^2}\tilde{u}_i^T\delta V\tilde{v}_i \tag{11}$$

where $\tilde{s}_i$ are diagonal elements of $\tilde{\mathbf{s}}$, $\tilde{u}_i$ and $\tilde{v}_i$ are column vectors of $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$, respectively.

3. Results

The performance of the proposed method (SVDp) is evaluated using numerical simulation and the results are compared with the conventional method (SVDc).

A circular domain can be used as a cross-section of an industrial process pipe. In this paper, the circular domain is used with 4 cm in radius and 1 cm in height. A fine mesh with 3104 elements is used for the computation of boundary voltage data, whereas a coarse mesh with 776 elements is used for the computation of the inverse solver. Adjacent currents with 10 mA amplitude are injected into the domain through 16 electrodes. To determine the regularization parameters for the SVDc and SVDp methods, the L-curve method [7], [9] is used.

For quantitative comparison of the inverse solvers in the reconstructed images, the image error (IE) and the correlation coefficient (CC) are used [10]

$$\mathrm{IE} = \frac{\|\sigma - \hat{\sigma}\|}{\|\sigma\|} \tag{12}$$

$$\mathrm{CC} = \frac{\sum_{i=1}^{N}\left[\left(\sigma_i - \bar{\sigma}\right)\left(\hat{\sigma}_i - \bar{\hat{\sigma}}\right)\right]}{\sqrt{\sum_{i=1}^{N}\left(\sigma_i - \bar{\sigma}\right)^2\sum_{i=1}^{N}\left(\hat{\sigma}_i - \bar{\hat{\sigma}}\right)^2}} \tag{13}$$

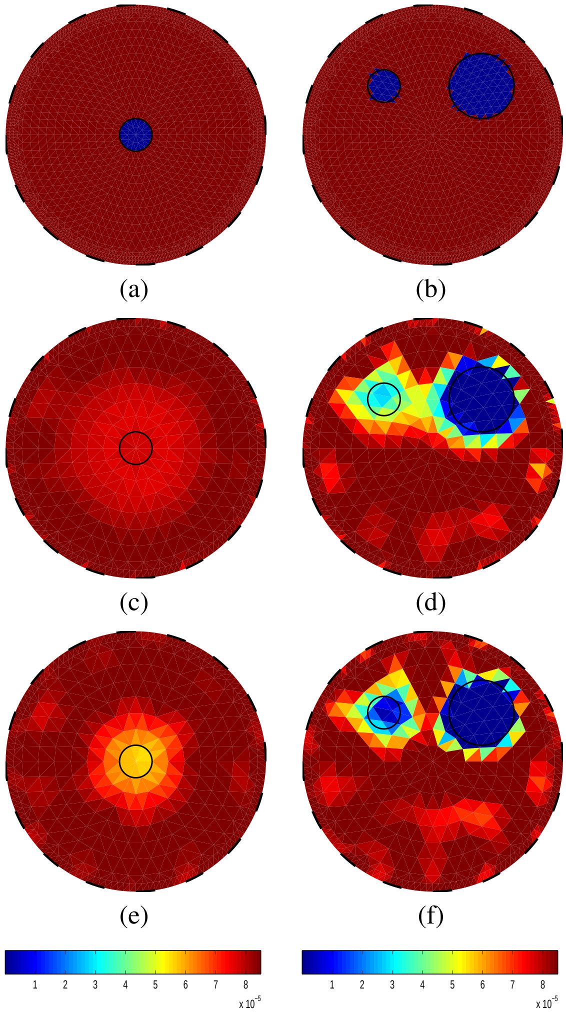

Figure 1. Reconstructed images: first row - true images, second row - images by SVDc and third row - images by SVDp. The black circles in the images represent the true positions of the anomalies.

where σ and $\hat{\sigma}$ are the true and estimated conductivity vectors, respectively, and $\bar{\sigma}$ and $\bar{\hat{\sigma}}$ are the mean values of σ and $\hat{\sigma}$, respectively. It is noticed that smaller IE and bigger CC values indicate better reconstruction performance of the inverse solver.

It is assumed that the background conductivity is 85×10^{-6} S/cm that is similar to the conductivity of tap water and the anomaly conductivity is 3×10^{-10} S/cm as bubbles or voids. For noisy measurements, a zero-mean Gaussian random noise is added to the calculated voltage data with 1% noise level.

Two scenarios are considered to verify the performance of the inverse solvers. In the first scenario, a single anomaly with 0.5 cm in radius is located at the center, as shown in Fig. 1(a). In the second scenario, two different sizes of the anomalies are located in the domain, as shown in Fig. 1(b). Moreover, all the reconstructions are displayed in the same color scales (3×10^{-10} – 85×10^{-6} S/cm) to compare the reconstructed images.

The true and reconstructed images are shown in Fig. 1. The

Table 1. Image error (IE) and correlation coefficient (CC) for the numerical scenarios.

	scenario 1		scenario 2	
	IE	CC	IE	CC
SVDc	0.1127	0.1932	0.1950	0.6954
SVDp	0.1019	0.4807	0.1943	0.7157

Table 2. Online/offline computational times for each method.

	SVDc	SVDp	GSVD
online	0.743 msec	0.566 msec	0.815 msec
offline	0.133 sec	1.968 sec	2.060 sec

first row shows the true images with single or multiple anomalies. The second and third rows present the reconstructed images by using the SVDc and SVDp methods, respectively. In the reconstructed images of Fig. 1, it is noted that the reconstructed images by the proposed SVDp method has better resolution than those of conventional SVDc method.

The IE and CC values are shown in Table 1. From Table 1, it is clear that the SVDp method has gives better reconstruction performance compared to the SVDc method.

The online computational times in the inverse calculation are shown in Table 2 (Computer used: Intel(R) Core(TM) i5-3570 CPU at 3.4 GHz, 8.0 GB RAM, Windows 7, Matlab version 7.1 (R14)). The SVDc, SVDp and GSVD methods take 0.743 msec, 0.566 mssec and 0.815 msec, respectively in the online calculation, whereas the SVDp and GSVD methods needs more time, i.e. approximately 1.968 sec and 2.060 sec, respectively in the offline calculation. Here, the computational time is the average computed after each method is run 10 times.

4. Conclusions

In this paper, an inverse method based on the SVD method is proposed to solve the EIT inverse problem with the generalized Tikhonov regularization to determine the conductivity distribution. In order to reduce the online computational time of the inverse solver, the generalized regularization matrix is taken out from the inverse computation term and the SVD method is applied. Numerical simulations have been carried out to validate the reconstruction performance of the proposed method. The results show that the proposed method estimates the conductivity distribution with better accuracy compared to the conventional SVD method.

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References

[1] O.C. Jones, J.T. Lin, L. Ovacik, H. Shu, “Impedance imaging relative to gas-liquid systems,” *Nucl. Eng. Des.*, vol. 141, pp.159-176, 1993.

[2] J.G. Webster, *Electrical Impedance Tomography*, Adam Hilger, 1990.

[3] D.S. Holder, *Electrical Impedance Tomography: Methods, History and Applications*, IOP Publishing, 2005.

[4] D.C. Barber, B.H. Brown, "Progress in electrical impedance tomography," ed. D. Colton, R. Ewing, W. Rundell, *Inverse Problems in Partial Differential Equations*, SIAM, pp.151-164, 1990.

[5] M. Cheney, D. Isaacson, J.C. Newell, S. Simske, J. Goble, "NOSER: An algorithm for solving the inverse conductivity problem," *Int. J. Imaging Syst. Technol.*, vol. 2, pp.66-75, 1990.

[6] J.L. Mueller, D. Isaacson, J.C. Newell, "A reconstruction algorithm for electrical impedance tomography data collected on rectangular electrode arrays," *IEEE Trans. Biomed. Eng.*, vol. 46, pp.1379-1386, 1999.

[7] P.C. Hansen, *Rank-Deficient and Discrete Ill-Posed Inverse Problems,* SIAM, 1998.

[8] M. Vauhkonen, W.R.B. Lionheart, L.M. Heikkinen, P.J. Vauhkonen, J.P. Kaipio, "A MATLAB package for the EIDORS project to reconstruct two-dimensional EIT images," *Physiol. Meas.*, vol. 22, pp.107-111, 2001.

[9] P.C. Hansen, D.P. O'Leary, "The use of the L-curve in the regularization of discrete ill-posed problems," *SIAM J. Sci. Comput.*, vol. 14, pp.1487-1503, 1993.

[10] W.Q. Yang, L. Peng, "Image reconstruction algorithms for electrical capacitance tomography," *Meas. Sci. Technol.*, vol. 14, pp.R1-13, 2003.