Preliminary Study of the Importance of Small Eigenvalues through Matrix Approximation without Rank Restriction

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1. Introduction

The structure of networks is often represented by the matrix (e.g., Laplacian matrix), and graph algorithms are designed using its top eigenvalues with large absolute values. The low-rank approximation proves the importance of the top eigenvalues in the graph algorithms since these eigenvalues contain more information than the bottom ones.

On the other hand, we clarified that the performance of a graph algorithm can be improved by using the combination of top eigenvalues and bottom eigenvalues [1]. This suggests that there are cases in which the bottom eigenvalues are also useful, but it is not clear when they are useful.

Therefore, in this paper, in order to conduct a preliminary study to understand when the bottom eigenvalues are useful, we discuss a matrix approximation without the rank restriction in the low-rank approximation.

2. Preliminary

Let $\boldsymbol{M}$ be a $n \times n$ matrix. $\boldsymbol{M}$ can be expressed as

$$\boldsymbol{M} = \sum_{k=1}^{n} \mu_k \boldsymbol{v}_k \boldsymbol{v}_k^{\mathsf{T}}. \tag{1}$$

where μ_k is the k-th smallest eigenvalue of $\boldsymbol{M}$, and $\boldsymbol{v}_k$ is the eigenvector corresponding to μ_k. Based on the low-rank approximation, we can obtain the rank-N approximated matrix that is closest to $\boldsymbol{M}$ by substituting $\mu_1 = \mu_2 = \cdots = \mu_{n-N} = 0$ into Eq. (1). Therefore, it is widely believed that the top eigenvalues (i.e., $\mu_{n-N+1}, \mu_{n-N+2}, \ldots, \mu_N$) contain more information than bottom ones.

3. Matrix Approximation without Rank Restriction

When the rank restriction in the low-rank approximation exists, we cannot discuss cases when the bottom eigenvalues are useful. Therefore, we focus on a matrix approximation relaxing the rank restriction. This matrix approximation is a generalized low-rank approximation. To approximate a matrix, we select N eigenvalues, and substitute the constant value c for the unselected eigenvalues. Namely, $\boldsymbol{M}$ is approximated by

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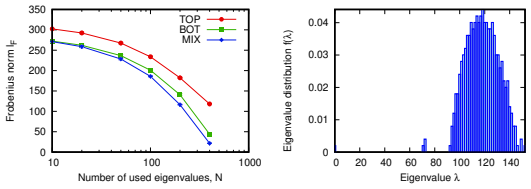

(a) $l_F(\phi_N)$ for different ϕ_N (b) Eigenvalue distribution of $\boldsymbol{L}$

Fig. 1: The results of numerical examples for the Laplacian matrix $\boldsymbol{L}$

$$\hat{\boldsymbol{M}}_{\phi_N} = \sum_{k\in\phi_N} \mu_k \boldsymbol{v}_k \boldsymbol{v}_k^{\mathsf{T}} + \sum_{k\in\tilde{\phi}_N} c\, \boldsymbol{v}_k \boldsymbol{v}_k^{\mathsf{T}} = c\boldsymbol{I} - \sum_{k\in\phi_N} (c-\mu_k) \boldsymbol{v}_k \boldsymbol{v}_k^{\mathsf{T}}, \tag{2}$$

where $\boldsymbol{I}$ is the $n \times n$ identity matrix. ϕ_N and $\tilde{\phi}_N$ are the index set of selected N eigenvalue, and the complement of ϕ_N. Our matrix approximation is equivalent to the low-rank approximation if $c = 0$ and $\phi_N = \{n-N+1, n-N+2, \cdots, n\}$ in Eq. (2). The approximation error (Frobenius norm) $l_F(\phi_N) = \|\boldsymbol{M} - \hat{\boldsymbol{M}}_{\phi_N}\|$ is given by $l_F(\phi_N) = \sqrt{\sum_{k\in\tilde{\phi}_N} (c-\mu_k)^2}$. When ϕ_N is selected, the optimal value of c to minimize $l_F(\phi_N)$ is uniquely determined by $c = \frac{1}{n-N}\left(\mathrm{Tr}(\boldsymbol{M}) - \sum_{k\in\phi_N} \mu_k\right)$. Therefore, $\hat{\boldsymbol{M}}_{\phi_N}$ can be constructed using only the eigenvalues μ_k and eigenvectors $\boldsymbol{v}_k$ for $\forall k \in \phi_N$, and $l_F(\phi_N)$ is useful to discuss the information amount contained in the selected eigenvalues in ϕ_N.

Figure 1 (a) shows the Frobenius norm $l_F(\phi_N)$ of the Laplacian matrix $\boldsymbol{L}$ for the popular random networks (stochastic block networks) when using different ϕ_N. *TOP*, *BOT* and *MIX* are the results selecting (a) top N eigenvalues, (b) bottom N eigenvalues, (c) top $N/2$ and bottom $N/2$ eigenvalues, respectively. According to the results, the Frobenius norm $l_F(\phi_N)$ for MIX is smallest. This means that not only the top and but also bottom eigenvalues contain a lot of information of $\boldsymbol{L}$. According to Fig. 1 (b), these eigenvalues in MIX are located at low densities of distribution $f(\mu)$ for $\boldsymbol{L}$. This implies the relationship between eigenvalue density and the information amount in the selected eigenvalues in ϕ_N.

For future work, we will utilize the bottom eigenvalues to construct new algorithms.

References

[1] E. Segawa and Y. Sakumoto, "Preliminary study for the impact of small eigenvalues on Laplacian anomaly detection of dynamic networks," Proceedings of COMPSAC 2022, pp.426–427, June 2022.