Radiation From An Axial Electric Dipole With Oblate Spheroidal Metamaterial Cloak Cover

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***Abstract*—Exact analytical expressions for the electromagnetic field due to an electric dipole source located inside and along the axis of symmetry of an oblate spheroid are derived. The spheroid is made of a penetrable, linear, isotropic, homogeneous and lossless material whose propagation constant is either equal (isorefractive cloak) or of opposite sign (anti-isorefractive metamaterial cloak) to the propagation constant of the medium surrounding the spheroid. In particular, the influence of the spheroidal material on the radiation pattern of the spheroid is examined.**

I. Introduction

The radiation from a dipole surrounded by an oblate spheroidal metamaterial cloak structure enclosure is analyzed. The dipole source is located on the axis of symmetry and axially oriented in a oblate spheroidal coordinate system, as shown in Fig. 1. The cloak is composed of a linear, homogeneous, isotropic and lossless double-positive (DPS) material or double-negative (DNG) metamaterial. The DPS material has positive values of permittivity ε, permeability μ, and wavenumber k while the DNG metamaterial is characterized by negative values of permittivity ε, permeability μ and wavenumber k. Electric and magnetic fields in each region are obtained in terms of infinite series containing oblate spheroidal functions, using the notation of Flammer [1]. In order to find an exact analytical solution of the problem, the mode matching technique requires isorefractive or anti-isorefractive conditions, implying that the absolute values of the wavenumbers in each region are the same, but the intrinsic impedances are different. The analysis is performed in the phasor domain with a time-dependence factor $\exp(j\omega t)$ that is omitted throughout this manuscript. This article provides the analytical solution and numerical results on the effects of cloaking on the directivity will be presented at the conference.

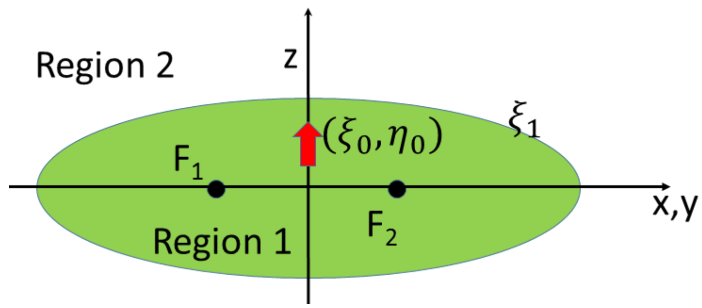

Fig. 1. Problem geometry.

This analysis is motivated by some applications where metamaterial cloaking structures have been investigated to reduce the scattering from receiving antennas [2], [3]. Additionally, from the point of view of transmitting antennas, a DNG spheroidal cloak structure improves the radiation power from a dipole, and resonant phenomenon gives also frequency dependent behavior [4]. Related geometrical structures have been investigated in [5], [6].

II. Electric dipole source

The geometry of the problem is shown in Fig. 1 and is best described in the oblate spheroidal coordinate system (η, ξ, φ), which is related to the rectangular coordinate system (x, y, z) through the relationships

$$x = \frac{d}{2}\sqrt{(\xi^2+1)(1-\eta^2)}\cos\varphi \tag{1}$$

$$y = \frac{d}{2}\sqrt{(\xi^2+1)(1-\eta^2)}\sin\varphi \tag{2}$$

$$z = \frac{d}{2}\xi\eta \tag{3}$$

where d is the focal distance, $-1 \leq \eta \leq 1$, $0 \leq \xi$, and $0 \leq \varphi \leq 2\pi$. Additional details on the oblate spheroidal system are provided, for example, in [1], [7], and an expression for the inverse relationship between cartesian and oblate spheroidal coordinates is given in [8].

A. Cloak made of isorefractive material

When regions 1 and 2 are isorefractive to each other, the propagation constants β_1 and β_2 are such that $\beta_1 = \beta_2 = \beta$. Consider an electric dipole located inside region 1 on the axis of symmetry at (ξ_0, η_0). Its incident electric Hertz vector is given by [7]

$$\mathbf{\Pi}^{\boldsymbol{i}} = \frac{e^{-j\beta R}}{\beta R}\hat{\mathbf{z}}, \tag{4}$$

where R is the distance between the dipole and the observation point. The magnetic field has only a $\hat{\varphi}$ directed component given by

$$\begin{aligned} H^i_{1\varphi} = & \frac{2\beta^2 Y_1}{\sqrt{\xi_0^2+1}} \sum_{n=1}^{\infty} \frac{j^n}{\tilde{\rho}_{1,n}\tilde{N}_{1,n}} \\ & \times R^{(1)}_{1,n}(-jc, j\xi_<) R^{(4)}_{1,n}(-jc, j\xi_>) S_{1,n}(-jc,\eta), \end{aligned} \tag{5}$$

where $c = \beta d/2$, $R^{(1)}_{1,n}$ and $R^{(4)}_{1,n}$ are radial oblate spheroidal functions, and $S_{1,n}$ are oblate angular spheroidal functions, according to the notation of Flammer [1]. The electric and magnetic fields inside regions 1 and 2 are given by

$$\begin{cases} \mathbf{E_h} = E_{h\xi}(\xi,\eta)\hat{\boldsymbol{\xi}} + E_{h\eta}(\xi,\eta)\hat{\boldsymbol{\eta}} \\ \mathbf{H_h} = H_{h\varphi}(\xi,\eta)\hat{\boldsymbol{\varphi}} \end{cases} \tag{6}$$

where $h = 1, 2$ depends on the region under consideration. The electric field components may be obtained from the magnetic field component $H_{h\varphi}$ using Maxwell's equations according to

$$E_{h\xi} = \frac{jZ_h}{c}\sqrt{\frac{1-\eta^2}{\xi^2+\eta^2}}\left(\frac{\partial}{\partial\eta} - \frac{\eta}{1-\eta^2}\right) H_{h\varphi} \tag{7}$$

$$E_{h\eta} = -\frac{jZ_h}{c}\sqrt{\frac{\xi^2+1}{\xi^2+\eta^2}}\left(\frac{\partial}{\partial\xi} + \frac{\xi}{\xi^2+1}\right) H_{h\varphi} \tag{8}$$

Inside region 1, there is also a scattered field given by

$$\begin{aligned} & H^s_{1\varphi} = \\ & \frac{2\beta^2 Y_1}{\sqrt{\xi_0^2+1}} \sum_{n=1}^{\infty} \frac{j^n}{\tilde{\rho}_{1,n}\tilde{N}_{1,n}} a_n R^{(1)}_{1,n}(-jc, j\xi) S_{1,n}(-jc,\eta) \end{aligned} \tag{9}$$

One should notice that the most general electromagnetic field inside region 1 requires the superposition of two linearly independent radial functions, i.e those of the first kind and those of the fourth kind. However, the functions of the fourth kind cannot be included because we seek a solution that is valid for all values of c, including the limiting case $c \to 0$. However, in such a case, the functions $R^{(4)}_{1,n}$ approach the spherical Hankel functions, which are singular at the origin. In the outer region 2, the total field is written to satisfy the radiation condition at infinity as

$$\begin{aligned} & H_{2\varphi} = \\ & \frac{2\beta^2 Y_2}{\sqrt{\xi_0^2+1}} \sum_{n=1}^{\infty} \frac{j^n}{\tilde{\rho}_{1,n}\tilde{N}_{1,n}} b_n R^{(4)}_{1,n}(-jc, j\xi) S_{1,n}(-jc,\eta). \end{aligned} \tag{10}$$

The tangential components of the electric field are

$$\begin{aligned} E_{1\eta} = & -\frac{jZ_1}{c}\frac{2\beta^2 Y_1}{\sqrt{\xi_0^2+1}}\sqrt{\frac{\xi^2+1}{\xi^2+\eta^2}} \sum_{n=1}^{\infty} \frac{j^n}{\tilde{\rho}_{1,n}\tilde{N}_{1,n}} \\ & \times \Big[a_n R^{(1)'}_{1,n}(-jc, j\xi_1) + R^{(1)}_{1,n}(-jc, j\xi_0) R^{(4)'}_{1,n}(-jc, j\xi_1) \\ & + \frac{\xi_1}{\xi_1^2+1}\Big(a_n R^{(1)}_{1,n}(-jc, j\xi_1) \\ & + R^{(1)}_{1,n}(-jc, j\xi_0) R^{(4)}_{1,n}(-jc, j\xi_1)\Big)\Big] S_{1,n}(-jc,\eta) \end{aligned} \tag{11}$$

$$\begin{aligned} E_{2\eta} = & -\frac{jZ_2}{c}\frac{2\beta^2 Y_2}{\sqrt{\xi_0^2+1}}\sqrt{\frac{\xi^2+1}{\xi^2+\eta^2}} \sum_{n=1}^{\infty} \frac{j^n}{\tilde{\rho}_{1,n}\tilde{N}_{1,n}} b_n \\ & \times \left[R^{(4)'}_{1,n}(-jc, \xi_1) + \frac{\xi_1}{\xi_1^2+1} R^{(4)}_{1,n}(-jc, j\xi_1)\right] S_{1,n}(-jc,\eta) \end{aligned} \tag{12}$$

The unknown expansion coefficients are determined by the application of the boundary conditions at the interface $\xi = \xi_1$ yielding

$$a_n = \frac{\Delta_a}{\Delta} \tag{13}$$

$$b_n = \frac{\Delta_b}{\Delta} \tag{14}$$

where

$$\begin{aligned} \Delta = & -Y_1 R^{(1)}_{1,n}(-jc, j\xi_1) \\ & \times \left(R^{(4)'}_{1,n}(-jc, j\xi_1) + \frac{\xi_1}{\xi_1^2+1} R^{(4)}_{1,n}(-jc, j\xi_1)\right) \\ & + Y_2 R^{(4)}_{1,n}(-jc, j\xi_1) \end{aligned}$$

$$\times\left(R_{1,n}^{(1)'}(-jc,j\xi_1)+\frac{\xi_1}{\xi_1^2+1}R_{1,n}^{(1)}(-jc,j\xi_1)\right) \quad (15)$$

$$\begin{aligned}\Delta_a &= Y_1 R_{1,n}^{(1)}(-jc,\xi_0)R_{1,n}^{(4)}(-jc,j\xi_1)\\ &\times\left(R_{1,n}^{(4)'}(-jc,\xi_1)+\frac{\xi_1}{\xi_1^2+1}R_{1,n}^{(4)}(-jc,\xi_1)\right)\\ &- Y_2 R_{1,n}^{(4)}(-jc,j\xi_1)R_{1,n}^{(1)}(-jc,j\xi_0)\\ &\times\left(R_{1,n}^{(4)'}(-jc,j\xi_1)+\frac{\xi_1}{\xi_1^2+1}R_{1,n}^{(4)}(-jc,j\xi_1)\right) \quad (16)\end{aligned}$$

$$\begin{aligned}\Delta_b &= -Y_1 R_{1,n}^{(1)}(-jc,j\xi_1)R_{1,n}^{(1)}(-jc,j\xi_0)\\ &\times\left(R_{1,n}^{(4)'}(-jc,j\xi_1)+\frac{\xi_1}{\xi_1^2+1}R_{1,n}^{(4)}(-jc,j\xi_1)\right)\\ &+ Y_1 R_{1,n}^{(1)}(-jc,j\xi_0)R_{1,n}^{(4)}(-jc,j\xi_1)\\ &\times\left(R_{1,n}^{(1)'}(-jc,j\xi_1)+\frac{\xi_1}{\xi_2+1}R_{1,n}^{(1)}(-jc,j\xi_1)\right) \quad (17)\end{aligned}$$

To obtain the asymptotic behavior of the magnetic field, we observe that when $\xi \to \infty$ then $c\xi \to \beta r$ and $\eta \to \cos\theta$, where (r,θ,φ) is a spherical coordinate system, yielding

$$H_{2\varphi} \sim \frac{e^{-j\beta r}}{\beta r}F(\cos\theta) \quad (18)$$

where

$$F(\cos\theta)=\frac{2j\beta^2 Y_2}{\sqrt{\xi_0^2+1}}\sum_{n=1}^{\infty}\frac{(-1)^n}{\tilde{\rho}_{1,n}\tilde{N}_{1,n}}b_n S_{1,n}(-jc,\cos\theta) \quad (19)$$

B. *Cloak made of anti-isorefractive material*

When regions 1 and 2 are anti-isorefractive to each other, the propagation constants β_1 and β_2 are such that $-\beta_1=\beta_2=\beta$. Similar to the previous case, the Hertz vector of an electric dipole located inside region 1 on the axis of symmetry at (ξ_0,η_0) is given by

$$\mathbf{\Pi}^{i}=-\frac{e^{j\beta R}}{\beta R}\hat{\mathbf{z}} \quad (20)$$

where the sign of the propagation constant is changed to account for the presence of the DNG material. The magnetic field has only a $\hat{\varphi}$ directed component given by

$$\begin{aligned}H_{1\varphi}^{i} &= \frac{2\beta^2 Y_1}{\sqrt{\xi_0^2+1}}\sum_{n=1}^{\infty}\frac{j^n}{\tilde{\rho}_{1,n}\tilde{N}_{1,n}}\\ &\times R_{1,n}^{(1)}(jc,j\xi_<)R_{1,n}^{(4)}(jc,j\xi_>)S_{1,n}(jc,\eta), \quad (21)\end{aligned}$$

where the oblate spheroidal functions depend upon the parameter $-c$ inside the DNG region. The electric and magnetic fields inside regions 1 and 2 are still given by eq. (6) and the relationships (7) and (8) are still valid. The scattered field inside region 1 is written as

$$H_{1\varphi}^{s}=\frac{2\beta^2 Y_1}{\sqrt{\xi_0^2+1}}\sum_{n=1}^{\infty}\frac{j^n}{\tilde{\rho}_{1,n}\tilde{N}_{1,n}}c_n R_{1,n}^{(1)}(jc,j\xi)S_{1,n}(jc,\eta) \quad (22)$$

and, in the outer region 2, the total field is written to satisfy the radiation condition at infinity as

$$\begin{aligned}&H_{2\varphi} =\\ &\frac{2\beta^2 Y_2}{\sqrt{\xi_0^2+1}}\sum_{n=1}^{\infty}\frac{j^n}{\tilde{\rho}_{1,n}\tilde{N}_{1,n}}d_n R_{1,n}^{(4)}(-jc,j\xi)S_{1,n}(-jc,\eta). \quad (23)\end{aligned}$$

The tangential components of the electric field are given by

$$\begin{aligned}E_{1\eta} &= \frac{jZ_1}{c}\frac{2\beta^2 Y_1}{\sqrt{\xi_0^2+1}}\sqrt{\frac{\xi^2+1}{\xi^2+\eta^2}}\sum_{n=1}^{\infty}\frac{j^n}{\tilde{\rho}_{1,n}\tilde{N}_{1,n}}\\ &\times\Big[c_n R_{1,n}^{(1)'}(jc,j\xi_1)+R_{1,n}^{(1)}(jc,j\xi_0)R_{1,n}^{(4)'}(jc,j\xi_1)\\ &+\frac{\xi_1}{\xi_1^2+1}\Big(c_n R_{1,n}^{(1)}(jc,j\xi_1)\\ &+\ R_{1,n}^{(1)}(jc,j\xi_0)R_{1,n}^{(4)}(jc,j\xi_1)\Big)\Big]S_{1,n}(jc,\eta) \quad (24)\end{aligned}$$

$$\begin{aligned}E_{2\eta} &= -\frac{jZ_2}{c}\frac{2\beta^2 Y_2}{\sqrt{\xi_0^2+1}}\sqrt{\frac{\xi^2+1}{\xi^2+\eta^2}}\sum_{n=1}^{\infty}\frac{j^n}{\tilde{\rho}_{1,n}\tilde{N}_{1,n}}\\ &\times d_n\left(R_{1,n}^{(4)'}(-jc,j\xi_1)+\frac{\xi_1}{\xi_1^2+1}R_{1,n}^{(4)}(-jc,j\xi_1)\right)\\ &\times S_{1,n}(-jc,\eta) \quad (25)\end{aligned}$$

The unknown expansion coefficients are determined by the application of the boundary conditions at the interface $\xi=\xi_1$, yielding

$$c_n=\frac{\Delta_c}{\Delta_2} \quad (26)$$

$$d_n=\frac{\Delta_d}{\Delta_2} \quad (27)$$

where

$$\begin{aligned}\Delta_2 &= Y_1 R_{1,n}^{(1)}(jc,j\xi_1)\\ &\times\left(R_{1,n}^{(4)'}(-jc,j\xi_1)+\frac{\xi_1}{\xi_1^2+1}R_{1,n}^{(4)}(-jc,j\xi_1)\right)\\ &+ Y_2 R_{1,n}^{(4)}(-jc,j\xi_1)\\ &\times\left(R_{1,n}^{(1)'}(jc,j\xi_1)+\frac{\xi_1}{\xi_1^2+1}R_{1,n}^{(1)}(jc,j\xi_1)\right) \quad (28)\end{aligned}$$

$$
\begin{aligned}
\Delta_c = & -Y_1 R_{1,n}^{(1)}(jc, j\xi_0) R_{1,n}^{(4)}(jc, j\xi_1) \\
& \times \left(R_{1,n}^{(4)'}(-jc, j\xi_1) + \frac{\xi_1}{\xi_1^2+1} R_{1,n}^{(4)}(-jc, j\xi_1) \right) \\
& - Y_2 R_{1,n}^{(4)}(-jc, j\xi_1) R_{1,n}^{(1)}(jc, \xi_0) \\
& \times \left(R_{1,n}^{(4)'}(jc, j\xi_1) + \frac{\xi_1}{\xi_1^2+1} R_{1,n}^{(4)}(jc, j\xi_1) \right) \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
\Delta_d = & -Y_1 R_{1,n}^{(1)}(jc, \xi_1) R_{1,n}^{(1)}(jc, j\xi_0) \\
& \times \left(R_{1,n}^{(4)'}(jc, j\xi_1) + \frac{\xi_1}{\xi_1^2+1} R_{1,n}^{(4)}(jc, j\xi_1) \right) \\
& + Y_1 R_{1,n}^{(1)}(jc, j\xi_0) R_{1,n}^{(4)}(jc, j\xi_1) \\
& \times \left(R_{1,n}^{(1)'}(jc, j\xi_1) + \frac{\xi_1}{\xi_1^2+1} R_{1,n}^{(1)}(jc, j\xi_1) \right) \qquad (30)
\end{aligned}
$$

Similar to the previous section, the asymptotic behavior of the magnetic field is given by

$$
H_{2\varphi} \sim \frac{e^{-j\beta r}}{\beta r} G(\cos\theta) \qquad (31)
$$

where

$$
G(\cos\theta) = \frac{2j\beta^2 Y_2}{\sqrt{\xi_0^2+1}} \sum_{n=1}^{\infty} \frac{(-1)^n}{\tilde{\rho}_{1,n}\tilde{N}_{1,n}} d_n S_{1,n}(-jc, \cos\theta) \qquad (32)
$$

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