Improvement of Metasurface Continuity Conditions

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***Abstract*—We analyse the limitations of an ideal zero-thickness sheet model, based on electromagnetic susceptibility tensors, to synthesize a sub-wavelength thick metasurface. First, the ideal zero-thickness model is used to synthesize an absorbing metasurface in terms of its susceptibilities. Then, we show the discrepancies between the response of the zero-thickness metasurface and the response of a sub-wavelength thin slab possessing the same electromagnetic susceptibilities. Finally, we derive higher order continuity conditions to provide a more rigorous treatment of the problem of sub-wavelength thick metasurfaces.**

I. Introduction

Metasurfaces [1]–[3] are dimensional reductions of volume metamaterials and functional extensions of frequency selective surfaces [4]. They are composed of two-dimensional arrays of sub-wavelength scattering particles engineered in such a manner that they transform incident waves into desired reflected and transmitted waves. Compared to volume metamaterials, metasurfaces offer the advantage of being lighter, easier to fabricate and less lossy due to their reduced dimensionality, while compared to frequency selective surfaces, they provide greater flexibility and functionalities.

In order to design and implement metasurfaces, it is convenient to consider metasurfaces as zero-thickness interfaces rather than sub-wavelength thick slabs because of the higher complexity in the resolution of multiple interface problems. Based on this consideration, several synthesis techniques have been developed [5]–[8]. These techniques provide the metasurface constitutive parameters for the specified incident, reflected and transmitted fields. Even though these constitutive parameters only apply to zero-thickness interfaces, they can be used to describe a sub-wavelength thick metasurface.

In this work, we analyse the discrepancies between the ideal zero-thickness model and the response of a sub-wavelength thick metasurface. Moreover, we derive boundary conditions of higher order that describe more accurately metasurfaces with substantial sub-wavelength thickness.

II. Metasurface Synthesis Technique

A zero-thickness metasurface, introducing discontinuities in the electromagnetic field, can be rigorously described using distribution theory, as demonstrated by Idemen [9]. The generalized sheet transition conditions (GSTCs) are rigorous boundary conditions for zero-thickness interfaces first derived by Idemen and later applied to metasurfaces by Kuester *et al.* [1]. For a metasurface lying at $z = 0$ in the $x - y$ plane, they may be written as

$$\hat{z} \times \Delta \boldsymbol{H} = j\omega \boldsymbol{P}_{\|} - \hat{z} \times \nabla_{\|} M_z, \tag{1a}$$

$$\Delta \boldsymbol{E} \times \hat{z} = j\omega\mu \boldsymbol{M}_{\|} - \nabla_{\|} \left(\frac{P_z}{\epsilon} \right) \times \hat{z}, \tag{1b}$$

$$\hat{z} \cdot \Delta \boldsymbol{D} = -\nabla \cdot \boldsymbol{P}_{\|}, \tag{1c}$$

$$\hat{z} \cdot \Delta \boldsymbol{B} = -\mu \nabla \cdot \boldsymbol{M}_{\|}, \tag{1d}$$

where $\boldsymbol{P}$ and $\boldsymbol{M}$ are the electric and magnetic polarization densities, respectively, and are given in terms of the arithmetic average of the fields on both sides of the metasurface. The operator Δ stands for the difference of the fields between both sides of the metasurface.

Relations (1a) and (1b) contain spatial derivatives applying on the normal components of $\boldsymbol{P}$ and $\boldsymbol{M}$. However, we will consider here metasurfaces only possessing in plane polarization densities, since such metasurfaces admit closed-form solutions of their susceptibilities [5]. In this case, using the general definitions of $\boldsymbol{P}$ and $\boldsymbol{M}$, relations (1a) and (1b) reduce to

$$\hat{z} \times \Delta \boldsymbol{H} = j\omega\epsilon \overline{\overline{\chi}}_{\text{ee}} \boldsymbol{E}_{\text{av}} + jk \overline{\overline{\chi}}_{\text{em}} \boldsymbol{H}_{\text{av}}, \tag{2a}$$

$$\Delta \boldsymbol{E} \times \hat{z} = j\omega\mu \overline{\overline{\chi}}_{\text{mm}} \boldsymbol{H}_{\text{av}} + jk \overline{\overline{\chi}}_{\text{me}} \boldsymbol{E}_{\text{av}}, \tag{2b}$$

where $\overline{\overline{\chi}}_{\text{ee}}$, $\overline{\overline{\chi}}_{\text{mm}}$, $\overline{\overline{\chi}}_{\text{em}}$ and $\overline{\overline{\chi}}_{\text{me}}$ are, respectively, the electric, magnetic, electromagnetic and magnetoelectric susceptibility tensors. The subscript "av" denotes the arithmetic average of the fields.

III. Description of the Problem

To illustrate the synthesis procedure and analyze the response of a sub-wavelength thick metasurface, a simple electromagnetic transformation is considered here, while more complex transformations will be addressed elsewhere. It consists in an absorbing metasurface that reduces the amplitude of a normally incident plane wave, as illustrated in Fig. 1. The reflected wave is specified to be zero ($\boldsymbol{E_r} = 0$) and the normally transmitted plane wave to exhibit a transmission coefficient T. The specified incident and transmitted electric fields are x-polarized and respectively given by $\boldsymbol{E_i} = e^{-jkz}\hat{\boldsymbol{x}}$ and $\boldsymbol{E_t} = Te^{-jkz}\hat{\boldsymbol{x}}$, where $0 \leqslant |T| \leqslant 1$. In such a simple transformation, the parameters $\overline{\overline{\chi}}_{\text{me}}$ and $\overline{\overline{\chi}}_{\text{em}}$ vanish and the susceptibilities $\overline{\overline{\chi}}_{\text{ee}}$ and $\overline{\overline{\chi}}_{\text{mm}}$ are diagonal tensors since no rotation of polarization is required [5]. Finally, after substitution and simplification, relations (2a) and (2b) yield the following electric and magnetic susceptibilities

$$\chi = \chi_{\text{ee}}^{xx} = \chi_{\text{mm}}^{yy} = \frac{2j}{k}\frac{(T-1)}{(T+1)}. \tag{3}$$

The susceptibilities in (3) can be easily converted into the electric permittivity, $\epsilon_r = 1 + \chi_{\text{ee}}^{xx}/d$ and the magnetic permeability, $\mu_r = 1 + \chi_{\text{mm}}^{yy}/d$, where d is the thickness of the metasurface [9]. Dividing by d dilutes the effect of the susceptibilities over the thickness of the metasurface. This is a valid approximation as long as d remains sub-wavelength [9].

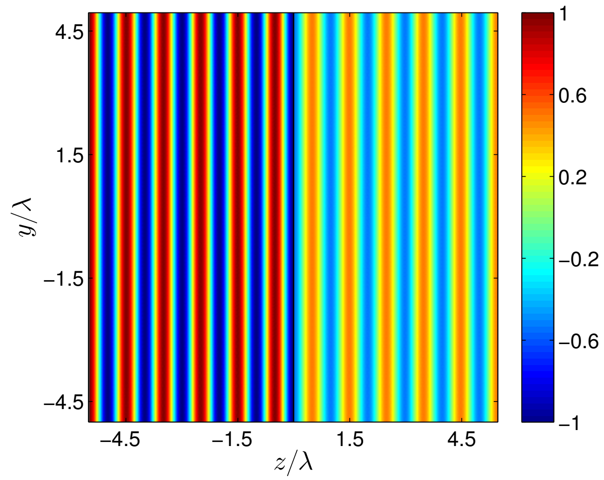

Fig. 1: Numerically simulated (COMSOL) $\text{Re}(E_x)$ field when a normally incident plane wave is impinging, from left to right, on a metasurface of thickness $d = \lambda/100$. The reflection-less metasurface is synthesize to reduce the amplitude of the incident wave by 50%.

Electromagnetic simulations are performed using COMSOL for different values of T and the results are reported in Fig. 2. As can be seen in the figure, for $T > 0.5$ the simulated transmission is in good agreement with the specification. In contrast, for $T < 0.5$ a discrepancy appears, that increases as T is reduced to 0. A more detailed analysis of these discrepancies will be provided elsewhere.

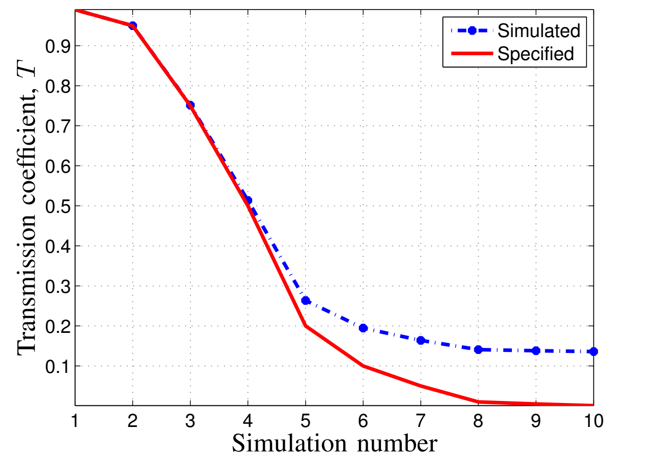

Fig. 2: Illustration of the discrepancies between the specified transmission (solid line) and the numerically simulated transmission (dashed line) from the metasurface of thickness $d = \lambda/100$.

One possibility, to reduce the aforementioned discrepancies, is to derive boundary conditions of higher orders than the GSTCs. As they stand in (1), the GSTCs only account for a zeroth order discontinuity, meaning that only the discontinuities of the fields are taken into account but not the discontinuities of the derivatives of the fields. Higher order boundary conditions that take into considerations the first derivative of the fields may therefore yield more accurate results. The developments of such boundary conditions are discussed in the following section.

IV. Derivation of Higher Order GSTCs

The GSTCs (1) can be derived by considering that all quantities (electric field, magnetic field, etc.) in Maxwell equations can be expressed, at the position of the metasurface, as [9]

$$f(z) = \{f(z)\} + \sum_{k=0}^{N} f_k \delta^{(k)}(z), \tag{4}$$

where $\{f(z)\}$ is the regular part of $f(z)$ and $\sum_{k=0}^{N} f_k \delta^{(k)}(z)$ is a Taylor-type series given in terms of the k^{th} derivative of the Dirac delta distribution. After substitution of (4) into Maxwell equations, two sets of equations are found, as shown in [5], [9]. The boundary conditions, for $k = 0$, are

$$\hat{z} \times \Delta\boldsymbol{H} + \nabla_{\parallel} \times \boldsymbol{H}_0 = j\omega\boldsymbol{D}_0, \tag{5a}$$

$$\hat{z} \times \Delta\boldsymbol{E} + \nabla_{\parallel} \times \boldsymbol{E}_0 = -j\omega\boldsymbol{B}_0, \tag{5b}$$

$$\hat{z} \cdot \Delta\boldsymbol{D} + \nabla_{\parallel} \cdot \boldsymbol{D}_0 = 0, \tag{5c}$$

$$\hat{z} \cdot \Delta\boldsymbol{B} + \nabla_{\parallel} \cdot \boldsymbol{B}_0 = 0, \tag{5d}$$

with the compatibility relations for $k \geqslant 1$

$$\hat{z} \times \boldsymbol{H}_{k-1} + \nabla_{\parallel} \times \boldsymbol{H}_k = j\omega\boldsymbol{D}_k, \tag{6a}$$

$$\hat{z} \times \boldsymbol{E}_{k-1} + \nabla_{\parallel} \times \boldsymbol{E}_k = -j\omega\boldsymbol{B}_k, \tag{6b}$$

$$\hat{z} \cdot \boldsymbol{D}_{k-1} + \nabla_{\parallel} \cdot \boldsymbol{D}_k = 0, \tag{6c}$$

$$\hat{z} \cdot \boldsymbol{B}_{k-1} + \nabla_{\parallel} \cdot \boldsymbol{B}_k = 0. \tag{6d}$$

The GSTCs are obtained by assuming that all terms are zero for $k \geqslant 1$. In that case, the electric field would be expressed as $\boldsymbol{E}(z) = \{\boldsymbol{E}(z)\} + \boldsymbol{E}_0(z)\delta(z)$ which takes into account only the discontinuity of the field but not of its derivatives. To obtain boundary conditions that take into account the first derivative of the fields, the series in (4) must be truncated at $k = 2$. Then by recursively solving relations (6), the boundary conditions are obtained as

$$\begin{aligned}\hat{z} \times \Delta\boldsymbol{H} =& j\omega\left(\hat{z} \times j\omega\epsilon\mu\boldsymbol{M}_{\mathbf{1},\parallel} - \hat{z} \times \nabla_{\parallel} \times P_{1,z} + \boldsymbol{P}_{\mathbf{0},\parallel}\right)\\ &- \hat{z} \times \nabla_{\parallel}\left(\nabla_{\parallel} \cdot \boldsymbol{M}_{\mathbf{1},\parallel} + M_{0,z}\right),\end{aligned} \tag{7a}$$

$$\begin{aligned}\hat{z} \times \Delta\boldsymbol{E} =& j\omega\left(\hat{z} \times j\omega\mu\boldsymbol{P}_{\mathbf{1},\parallel} + \hat{z} \times \nabla_{\parallel} \times \mu M_{1,z} - \mu\boldsymbol{M}_{\mathbf{0},\parallel}\right)\\ &- \hat{z} \times \nabla_{\parallel}\frac{1}{\epsilon}\left(\nabla_{\parallel} \cdot \boldsymbol{P}_{\mathbf{1},\parallel} + P_{0,z}\right),\end{aligned} \tag{7b}$$

$$\hat{z} \cdot \Delta\boldsymbol{D} = -\nabla_{\parallel} \cdot \left(\hat{z} \times j\omega\epsilon\mu\boldsymbol{M}_{\mathbf{1},\parallel} - \hat{z} \times \nabla_{\parallel} \times P_{1,z} + \boldsymbol{P}_{\mathbf{0},\parallel}\right), \tag{7c}$$

$$\hat{z} \cdot \Delta\boldsymbol{B} = -\mu\nabla_{\parallel} \cdot \left(\boldsymbol{M}_{\mathbf{0},\parallel} - \hat{z} \times j\omega\boldsymbol{P}_{\mathbf{1},\parallel} - \hat{z} \times \nabla_{\parallel} \times M_{1,z}\right). \tag{7d}$$

Relations (7) are the boundary conditions of first order. They can be compared to the GSTCs of zeroth order given in (1). One can easily verify that if all terms with subscript 1 (corresponding to $k = 1$) are dropped, relations (7) will reduce to (1). Similarly to what has been done to obtain relations (2), the normal component of $\boldsymbol{P}$ and $\boldsymbol{M}$ can be dropped yielding boundary conditions expressed only in terms of surface polarization densities

$$\hat{z} \times \Delta\boldsymbol{H} = j\omega\left(\hat{z} \times j\omega\epsilon\mu\boldsymbol{M}_{\mathbf{1},\parallel} + \boldsymbol{P}_{\mathbf{0},\parallel}\right) - \hat{z} \times \nabla_{\parallel}\left(\nabla_{\parallel} \cdot \boldsymbol{M}_{\mathbf{1},\parallel}\right), \tag{8a}$$

$$\hat{z} \times \Delta\boldsymbol{E} = j\omega\left(\hat{z} \times j\omega\mu\boldsymbol{P}_{\mathbf{1},\parallel} - \mu\boldsymbol{M}_{\mathbf{0},\parallel}\right) - \hat{z} \times \nabla_{\parallel}\frac{1}{\epsilon}\left(\nabla_{\parallel} \cdot \boldsymbol{P}_{\mathbf{1},\parallel}\right), \tag{8b}$$

$$\hat{z} \cdot \Delta\boldsymbol{D} = -\nabla_{\parallel} \cdot \left(\hat{z} \times j\omega\epsilon\mu\boldsymbol{M}_{\mathbf{1},\parallel} + \boldsymbol{P}_{\mathbf{0},\parallel}\right), \tag{8c}$$

$$\hat{z} \cdot \Delta\boldsymbol{B} = -\mu\nabla_{\parallel} \cdot \left(\boldsymbol{M}_{\mathbf{0},\parallel} - \hat{z} \times j\omega\boldsymbol{P}_{\mathbf{1},\parallel}\right). \tag{8d}$$

Compared to (2), the last term of relations (8a) and (8b) are spatial derivatives. Meaning that solving (8) may be more involved than the usual GSTCs. Future investigations will confirm that these new relations are more accurate and therefore worth considering despite their increased complexity.

V. Conclusion

By considering a simple electromagnetic transformation, we have shown how the GSTCs can be used to synthesize a zero-thickness metasurface and find its constitutive parameters. Then, these parameters have been applied to a thin material slab and the discrepancies between the response of the model and the material slab have been presented. We noted that the more demanding is the specified transformation (e.g. perfect absorption), the larger is the error. Finally, the GSTCs have been extended to take into account higher order terms allowing a better description of the metasurface.

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