A Closed Form BER expression for an Overlap-based CSS System

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Abstract: Overlap is one of the techniques for increasing bit rate in CSS. More overlaps can offer higher data throughput; however, they may cause more intersymbol interference (ISI) at the same time, resulting in serious bit error rate (BER) performance degradation. This implies that the number of overlaps should be decided according to the required system performance. In this paper, we derive a closed form expression for BER of the overlap-based CSS system, which would be very helpful in setting the number of overlaps. The numerical results demonstrate that the BER derived in a closed form closely agrees with the simulated BER.

1. Introduction

Chirp spread spectrum (CSS) is a spread spectrum technique that uses chirp signals and the associated pulse compression method. Since 1940s, CSS technology has been rapidly developed and deployed for military radar systems. The first suggestion for an application of chirp signals to communication systems was offered by Winkler in 1962. Main features of CSS technology such as high processing gain, high time resolution, low power consumption, anti-jamming, and anti-multipath fading make CSS suitable for indoor wireless communication systems [1]. In March 2007, IEEE has approved CSS physical layer (PHY) in its new wireless standard 802.15.4a. The new standard allows CSS to be used in applications such as real time location systems (RTLS), industrial control, sensor networking, and medical devices [2].

In CSS, a data signal is spread over a wider frequency band for transmission via chirp signals and the associated pulse compression method. CSS can be classified into two categories, depending on how to modulate chirp signals with data: one is binary orthogonal keying (BOK) and the other is direct modulation (DM). BOK uses chirp signals for representing data: for example, bits '1' and '0' can be represented by up-chirp and down-chirp signals, respectively. On the other hand, DM uses chirp signal just as a spreading code and performs data modulation and demodulation separately and independently from the chirp processing. Thus, DM allows various modulation techniques to be employed [3].

Overlap is one of the techniques for increasing bit rate in CSS. More overlaps can offer higher data throughput; however, they may cause more intersymbol interference (ISI) at the same time, resulting in serious bit error rate (BER) performance degradation [1]. Thus, the number of overlaps should be set adequately according to the required system performance, and in doing so, it would be very useful if there is a closed form expression of BER according to the number of overlaps. In [3], it was mentioned that BER performance changes according to the number of overlaps; however, no mathematical analysis or closed form associated with BER is not presented.

In this paper, thus, we are to derive a closed form expression for BER of the overlap-based CSS systems, exploiting the approximated Gaussian Q function. The remainder of this paper is organized as follows. In Section 2, basic CSS system model is described. Section 3 derives a closed form BER expression of an overlap-based system, and suggests parameter values to make the accuracy of the approximated Gaussian Q function higher. In Section 4, the analytic results are verified by BER performance simulation and finally Section 5 concludes this paper.

2. System model

The complex baseband equivalent $s(t)$ of a typical linear chirp waveform is expressed as

$$s(t) = \sqrt{\frac{E_b}{T_c}} \exp(j\pi\mu t^2), \quad |t| < \frac{T_c}{2}, \tag{1}$$

where E_b, T_c and μ denote the transmitted signal energy per bit, chirp duration, and chirp rate, respectively. The chirp rate is defined as the change rate of an instantaneous frequency. A chirp signal is called the up-chirp when μ is positive and the down-chirp when μ is negative. We consider binary phase shift keying (BPSK) system with the up-chirp. Then, we can express the i-th DM-BPSK chirp symbol $s_i(t)$ and the impulse response $h(t)$ of a filter matched to $s_i(t)$ as

$$s_i(t) = b_i s(t) \tag{2}$$

and

$$h(t) = A\sqrt{\frac{1}{T_c}} \exp(-j\pi\mu t^2), \tag{3}$$

respectively. In (2) and (3), $b_i \in \{\pm 1\}$ and A denote the i-th transmitted bit and a filter coefficient, respectively. Then the i-th matched filter output $g_i(t)$ is given by [4]

$$g_i(t) = s_i(t) * h(t)$$
$$= b_i(A\sqrt{E_b}) \frac{\sin\left\{\pi B t\left(1 - \frac{|t|}{T_c}\right)\right\}}{\pi B t}, \quad |t| < T_c, \tag{4}$$

where B $(=|\mu|T_c)$ is the chirp bandwidth defined as the range of the instantaneous frequency and '*' denotes the convolution operation.

3. Closed form BER expression

In this paper, we only consider an additive background noise as the channel impairment. If we transmit the overlapped chirp signal, the received signal is also overlapped, resulting in the overlapped matched filter output signal. For notational simplicity, first let us represent $\sin\{\pi Bt(1-|t|/T_c)\}/\pi Bt$ by $p(t)$. Then, we can express the i-th matched filter output $g_i^{overlap}(t)$ of the overlapped chirp signal sampled at t=0 as

$$g_i^{overlap}(t)\Big|_{t=0} = \sum_{k=-(O_f-1)}^{(O_f-1)} g_{i+k}(t-k\tau)+n\Big|_{t=0}$$
$$= (A\sqrt{E_b})\sum_{k=-(O_f-1)}^{(O_f-1)} b_{i+k}\,p(-k\tau)+n\,, \quad (5)$$

where O_f, τ $(=T_c/O_f)$, and n denote the total number of overlaps, symbol interval, and a Gaussian distributed noise component with mean zero and two-sided power spectral density $N_0/2$. Since b_i is identically distributed and Gaussian probability density function is symmetry about zero, we can express the probability of bit error P_B as

$$P_B = \frac{1}{2}\left\{\Pr(g_i^{overlap}(0)<0\,|\,b_0=1)+\Pr(g_i^{overlap}(0)>0\,|\,b_0=-1)\right\}$$
$$= \Pr(g_i^{overlap}(0)>0\,|\,b_0=-1)\,. \quad (6)$$

Denoting the i-th total normalized ISI z_i as

$$z_i = \sum_{\substack{k=-(O_f-1)\\ k\neq 0}}^{(O_f-1)} z_k\,, \quad (7)$$

where z_k $(=b_{i+k}\,p(-k\tau))$ denotes the k-th ISI component. Using (5), (6), and (7), we can express the probability of bit error conditioned on z_i as

$$P_{B|z_i} = Q\left(\sqrt{\frac{2E_b}{N_0}}(1-z_i)\right), \quad (8)$$

where $Q(x)=\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}dt$ is the Gaussian Q function.

Thus, the unconditioned probability of bit error can be obtained as

$$P_B = \mathrm{E}_{z_i}\left[Q\left(\sqrt{\frac{2E_b}{N_0}}(1-z_i)\right)\right], \quad (9)$$

where $\mathrm{E}_{z_i}\{\cdot\}$ denotes the expectation over z_i. To derive a closed form expression for P_B, we employ the following approximation (denoted by $\hat{Q}$) for the Gaussian Q function, which can be obtained based on series of exponentially decreasing cosine (EDC) [5]:

$$\hat{Q}(x) = \sum_{m=0}^{N_T-1} c_m e^{\lambda_m x}\cos(w_m x) = \mathrm{Re}\left[\sum_{m=0}^{N_T-1} c_m e^{(\lambda_m+jw_m)x}\right], \quad (10)$$

where parameters c_m, λ_m, w_m, and N_T are obtained subject to the constraint that the error between $Q(x)$ and $\hat{Q}(x)$ is minimized. After the substitution of (7) and (10) into (9) and some manipulations, we can rewrite the P_B as (11), where $M_{z_k}(t)$ $(=E\{e^{tz_k}\})$ is the moment generating function (MGF). Since $z_k=\{\pm p(-kT_c/O_f)\}$, $M_{z_k}(t)$ can be represented as

$$M_{z_k}(t) = \frac{1}{2}\left\{e^{tp\left(-\frac{kT_c}{O_f}\right)}+e^{-tp\left(-\frac{kT_c}{O_f}\right)}\right\} = \cosh\left\{p\left(\frac{-kT_c}{O_f}\right)t\right\}. \quad (12)$$

After the substitution of (12) into (11), finally, we get a closed form BER expression of the DM-BPSK system which is shown in the bottom of this page.

For $M>2$, the probability of bit error for the gray coded M-ary PSK $P_B(M)$ is given by [6]

$$P_B(M) \simeq \frac{2}{\log_2 M} Q\left\{\sqrt{\frac{2E_b(\log_2 M)}{N_0}}\sin\left(\frac{\pi}{M}\right)\right\}. \quad (14)$$

Using (14), we can extend the closed form BER expression obtained for BPSK to M-ary PSK as given by (15).

In this paper, we set BER range of our interest from 0.5 to 10^{-10}. The parameters c_m, λ_m, w_m, and N_T for guaranteeing an accuracy of the derived closed form BER expression over the target BER range are obtained subject to the constraint that the symmetric squared relative error (SSRE) ε between $Q(x)$ and $\hat{Q}(x)$ is minimized [7].

$$\varepsilon = \int_\chi \left[1-\hat{Q}(x)/Q(x)\right]^2+\left[1-Q(x)/\hat{Q}(x)\right]^2 dx$$

$$P_B \simeq \mathrm{Re}\left[\sum_{m=0}^{N_T-1} c_m e^{(\lambda_m+jw_m)\sqrt{\frac{2E_b}{N_0}}} \prod_{\substack{k=-(O_f-1)\\ k\neq 0}}^{(O_f-1)} M_{z_k}\left\{-(\lambda_m+jw_m)\sqrt{\frac{2E_b}{N_0}}\right\}\right]. \quad (11)$$

$$P_B \simeq \mathrm{Re}\left[\sum_{m=0}^{N_T-1} c_m e^{(\lambda_m+jw_m)\sqrt{\frac{2E_b}{N_0}}} \prod_{\substack{k=-(O_f-1)\\ k\neq 0}}^{(O_f-1)} \cosh\left\{p\left(-k\frac{T_c}{O_f}\right)(\lambda_m+jw_m)\sqrt{\frac{2E_b}{N_0}}\right\}\right]. \quad (13)$$

$$P_B \simeq \mathrm{Re}\left[\frac{2}{\log_2 M}\sum_{m=0}^{N_T-1} c_m e^{(\lambda_m+jw_m)\left(\sqrt{\frac{2(\log_2 M)E_b}{N_0}}\sin\frac{\pi}{M}\right)} \prod_{\substack{k=-(O_f-1)\\ k\neq 0}}^{(O_f-1)} \cosh\left\{p\left(-k\frac{T_c}{O_f}\right)(\lambda_m+jw_m)\left(\sqrt{\frac{2(\log_2 M)E_b}{N_0}}\sin\frac{\pi}{M}\right)\right\}\right]. \quad (15)$$

$$\simeq \frac{1}{N_s}\sum_{i=1}^{N_s}\left[1-\hat{Q}(x_i)/Q(x_i)\right]^2+\left[1-Q(x_i)/\hat{Q}(x_i)\right]^2, \quad (16)$$

where the formula in the second low of (16) is the discrete expression of the integral form in the first row and χ and N_s represent the range of the argument x over which we wish to minimize ε and the total number of samples over χ, respectively. Meanwhile, due to the fast decay rate of the Gaussian Q function, the wider BER range dealt with is, the longer EDC series and more parameters are required in order to guarantee the accuracy of P_B over the whole BER range. To solve this problem, we divide the entire BER range into two segments, the first one is set from 0.5 to 10^{-3} and the second one is set from 10^{-3} to 10^{-10}, respectively. Then SSRE is minimized numerically over the corresponding interval χ, where 1000 samples is obtained uniformly (N_s=1000) for each segment. Using these two individual EDC series for each BER segment, we can cover the entire BER range of our interest without long EDC series, after all. The parameters of $\hat{Q}(x)$ for each BER segment are obtained as shown in Tables 1 and 2.

Table 1. Parameters for $\hat{Q}(x)$ in the BER range from 0.5 to 10^{-3} ($N_T = 3$)

m	c_m	λ_m	w_m
0	- 5.498493	-2.394775	1.155583×10^{-1}
1	4.165956	-2.132909	3.038220×10^{-1}
2	1.483053	-1.957114	-4.425860×10^{-1}

Table 2. Parameters for $\hat{Q}(x)$ in the BER range from 10^{-3} to 10^{-10} ($N_T = 7$)

m	c_m	λ_m	w_m
0	-6.800443×10^{-1}	-2.639689	1.020882
1	-3.237126×10^{-1}	-2.696018	2.670366×10^{-2}
2	-1.595907×10^{-2}	-2.110599	-1.528960×10^{-1}
3	-7.330517	-3.314070	-1.167062
4	1.984192	-2.960136	1.449151×10^{-1}
5	2.390374	-2.818848	7.848084×10^{-3}
6	-1.639812	-2.919649	1.453802

4. Simulation results

In this section, we compare the theoretical BER of the overlap-based CSS systems derived in a closed form in this paper with the empirical BER. We assume that the sampling time synchronization is perfect. The evaluation conditions are as follow: $T_c = 0.5\mu s$, $\mu = 400\text{MHz}/\mu s$, and $B = 200\text{MHz}$. For $\hat{Q}(x)$, the parameter values represented in Tables 1 and 2 are used.

In Fig. 1, we can observe that when there is no overlap ($O_f = 1$), the theoretical BER for the gray coded DM-MPSK based on $\hat{Q}(x)$ is exactly same as the BER for the gray coded M-ary PSK signalling. Data modulation and

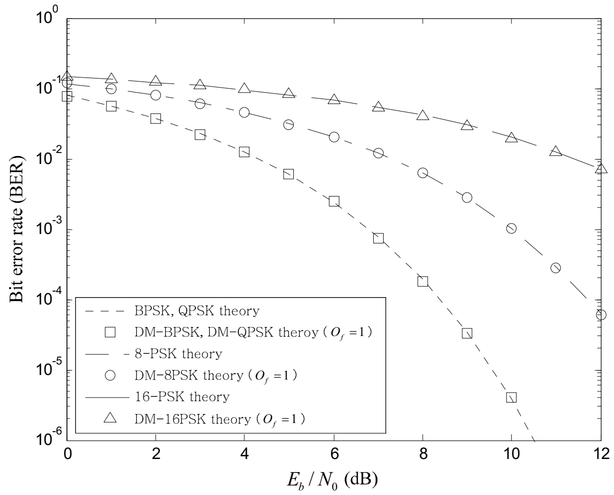

Fig. 1. Theoretical BER performance comparison between M-ary PSK and DM-MPSK system in an AWGN channel

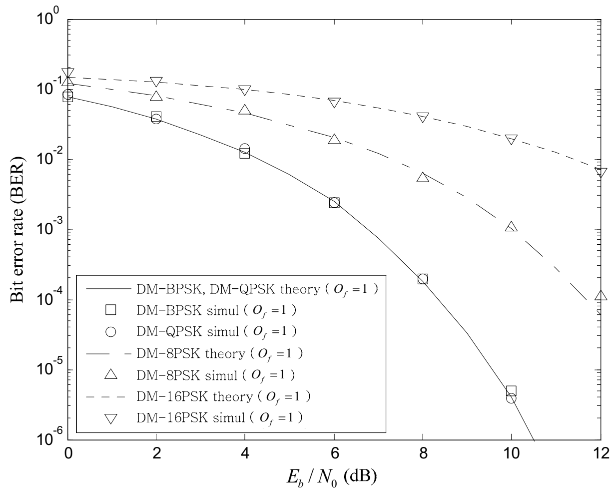

Fig. 2. Theoretical and simulated BER performance of the DM-MPSK system in an AWGN channel ($O_f = 1$)

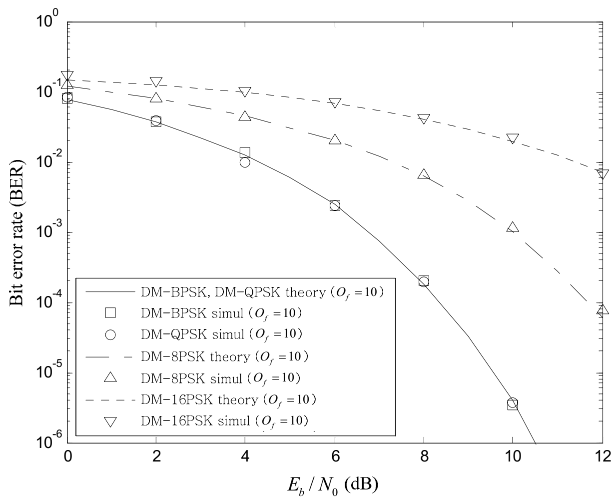

Fig. 3. Theoretical and simulated BER performance of the DM-MPSK system in an AWGN channel ($O_f = 10$)

demodulation are performed separately and independently from the chirp processing in DM. In other words, the pulse compression does not change the theoretical BER performance for an AWGN channel. Thus, if we do not perform the symbol overlap (and consequently, no ISI is occurred), the BER performance for DM-MPSK system is exactly same as that for M-ary PSK in an AWGN channel.

In Figs. 2 and 3, we can clearly see a close agreement between the theoretical BER based on $\hat{Q}(x)$ and simulated BER based on Monte Carlo runs. From the discussion in Section 3, we can easily anticipate that the difference between the theoretical and simulated BERs, becomes smaller as the value of N_T becomes larger. Another important observation is that there is little BER performance difference between cases that $O_f = 1$ and $O_f = 10$. This stems from the fact that the matched filter output in (4) is a sinc-like function having a very narrow mainlobe width. Even if several matched filter outputs are overlapped, there will be little ISI in the matched filter output of interest when symbol interval is much wider than mainlobe width of the matched filter output. When $O_f = 10$, the interval between chirp symbols is 50ns, which is about 5 times wider than the mainlobe width of the matched filter output of $2/B$. Therefore, BER performance degradation due to ISI is not serous when $O_f = 10$.

5. Conclusion

In this paper, we have derived the closed form expression for BER of the overlap-based CSS systems in an AWGN channel. Two individual EDC series with appropriate parameter values have been obtained to make the accuracy of the approximated Gaussian Q function higher. The numerical results have shown that the BER derived in a closed form closely agrees with simulated BER. The closed form BER expression derived in this paper should be very useful in deciding the number of overlaps according to a required system performance in the overlap-based CSS systems.

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References

[1] P. Zhang and H. Liu, "An ultra-wide band system with chirp spread spectrum transmission technique," in *Proc. Int. Conf. Intelligent Tutoring Systems (ITS),* pp. 294-297, Jhongli, Taiwan, June 2006.

[2] IEEE 802.15 WPAN Low Rate Alternative PHY TG4a. (March 2007), retrieved Apr. 2008, from http://www.ieee802.org/15/pub/TG4a.html

[3] J. Pinkley, *Low Complexity Indoor Wireless Data Links Using Chirp spread spectrum*, PhD thesis, Dept. Elect. Comm., University of Calgary, Calgary, AB, Canada, 2003.

[4] A. Springer, W. Gugler, R. Koller, and R. Weigel, "A wireless spread-spectrum communication system using SAW chirp delay lines," *IEEE Trans. Microwave theory and Techniques*, vol. 49, no. 4, pp. 754-760, Apr. 2001.

[5] O. Fonseca and I. N. Psaromiligkos, "BER performance of BPSK transmissions over multipath channels," *IET Electronics Letters*, vol. 42, no. 20, pp. 1164-1165, Sep. 2006.

[6] J. G. Proakis, *DigitalCommunications.*, New York, NY: McGraw-Hill, 2001.

[7] O. Fonseca and I. N. Psaromiligkos, "Approximation of the bit-error-rate of BPSK transmissions over frequency selective channels," in *Proc. IEEE Int. Conf. Wireless and Mobile Computing, Networking and Commun. (WiMob),* pp. 217-222, Montreal, QB, Canada, Apr. 2001.