Level Crossing Rate of Product of Nakagami-m Random Variable, Rician Random Variable and Rayleigh Random Variable

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***Abstract*—In this paper, product of three random variables will be considered. Level crossing rate (LCR) of product of Nakagami-m random variable, Rician random variable and Rayleigh random variable will be calculated. Obtained result can be used for evaluation the LCR of product of three Rayleigh random variables, LCR of product of two Rayleigh random variables and Rician random variable, and LCR of product of Nakagami-m random variable and two Rayleigh random variables.**

Keywords— ***Level crossing rate, Nakagami-m random variable, Rician random variable, Rayleigh random variable***

I. Introduction

In this paper, product of Nakagami-m random variable (RV), Rician random variable and Rayleigh random variable will be analyzed. Probability density function (PDF) and cumulative distribution function (CDF) will be calculated. By using PDF, the bit error probability (BEP) can be evaluated, and by using CDF, the outage probability (OP) can be obtained [1][2]. The outage probability and the bit error probability are the first order performance measure of wireless communication system. Level crossing rate (LCR) is the second order performance measure of wireless communication system, as well as average fade duration (AFD). AFD can be evaluated as the ratio of the OP and the LCR. The OP can be calculated as probability that signal envelope is below the threshold [3].

Obtained results have application in performance analysis of multi-hop relay wireless telecommunication systems when the signal level is much higher than the noise level. In such case, the noise level can be ignored. For that matter, the output signal is product of as many random variables as the sections in the relay system [4].

Compared to sums of random variables, interest for products started in the 1970s [5]-[6], but has been developing intensely over the past few years. Still, products of random variables arise naturally in many applications such as: channel modeling, multihop wireless relaying systems, cascaded fading channels, MIMO keyhole systems [7], quantum physics, signal processing, tensor sensing problem, the rate offset of the hybrid automatic repeat request (H-ARQ) transmission, and even in biological and physical sciences, econometrics, classification, ranking and selection [8].

The signal envelope variations, called fading, are results of reflections, refractions, diffraction and scattering. They can be described by several distributions. So, Rayleigh [9] and Nakagami-*m* [10] distributions are used when dominant component is not present. Signal envelope variation is modeled by Rician distribution when line-of sight (LOS) dominant component exists in the channel [11].

We examine here the scenario of the wireless relay communication system with three sections. In the first section, Nakagami-m fading is present, in the second section, Rician fading exists, and there is Rayleigh fading in the third section. The signal envelope at the output of relay communication system with three sections is product of Nakagami-*m* signal envelope, Rician signal envelope, and Rayleigh signal envelope. By using transformation method, PDF of product of Nakagami-m RV, Rician RV and Rayleigh RV will be obtained. By using PDF, CDF and moments can be evaluated. The first moment or the average value, the second moment or the square average value, and the third moment are important characteristic of wireless relay communication systems.

There are more works considering products and ratios of diferent random variables [5]-[8], [12]-[17]. For example in [14], LCR of product of *N* Rayleigh RVs is investigated. In this paper, AFD of wireless communication system operating over multipath Rayleigh fading is determined. Product of two Nakagami-m RVs is considered in [15]. LCR of product of two Nakagami-m variables is calculated and by using this formula, AFD of wireless relay communication system working over multipath Nakagami-m fading is evaluated. In that paper, RV equal to product of two Nakagami-*m* RVs is formed. This variable is known as Nakagami-*m**Nakagami-*m* RV.

The PDF, CDF, and moments functions of the *N*∗Nakagami distribution are developed in closed forms using the Meijer's G-function in [16]. Further, closed-form expressions for the system first order performance (OP, amount of fading, and average BEP) for several binary and multilevel modulation schemes working over the *N*∗Nakagami fading channel in the presence of Gaussian noise are derived. This is useful for designing of cascaded Nakagami-*m* fading channels. Further, in [17], PDF of the product of Rayleigh, exponentially, Nakagami-*m* and Gamma RVs is derived in closed form by the Mellin transform. In some papers, the LCR of ratio of product of two RVs and RV is presented, as well as the LCR of ratio of RV and product of two RVs.

In this work, radio relay system with three sections will be analyzed. For relay systems is very important to evaluate ratios and products of random variables. The paper is organized in four sections. After introduction, where previous works from this field are described, in section II, the derivation of the LCR of product of three RVs is presented. In third section, the validity of the theoretical results is confirmed by simulation results and parameters influence is analyzed. The last section is conclusion.

II. Level Crossing Rate of Product of Three Random Variables

Nakagami-m random variable x_1 follows distribution defined in [10]:

$$p_{x_1}(x_1)=\frac{2}{\Gamma(m)}\left(\frac{m}{\Omega_1}\right)^m x_1^{2m-1}e^{-\frac{m}{\Omega_1}x_1^2},\ x_1\geq 0 \tag{1}$$

where Ω_1 is power of x_1 and m is fading parameter of x_3.

Random variable x_2 follows Rician distribution [11]:

$$p_{x_2}(x_2)=\frac{2(\kappa+1)}{\Omega_2}\sum_{i_1=0}^{\infty}\left(\frac{\kappa(\kappa+1)}{\Omega_2}\right)^{i_1}\frac{1}{(i_1!)^2}x_2^{2i_1+1}e^{-\frac{\kappa+1}{\Omega_2}x_2^2},\ x_2\geq 0, \tag{2}$$

wherein Ω_2 is power of x_2 and κ is Rician factor. Rician factor is defined as ratio of dominant component power and scattering components powers.

Random variable x_3 follows Rayleigh distribution [1]:

$$p_{x_3}(x_3)=\frac{x_3}{\Omega_3}e^{-\frac{x_3^2}{\Omega_3}},\ x_3\geq 0, \tag{3}$$

and Ω_3 is power of x_3.

Product of x_1, x_2 and x_3 is:

$$x=x_1x_2x_3. \tag{4}$$

Then, it is valid:

$$x_1=\frac{x}{x_2x_3}. \tag{5}$$

The first derivative of x is:

$$\dot{x}=\dot{x}_1x_2x_3+x_1\dot{x}_2x_3+x_1x_2\dot{x}_3. \tag{6}$$

Random variables $\dot{x}_1$, $\dot{x}_2$ and $\dot{x}_3$ have Gaussian distribution. Linear combination of Gaussian RVs is Gaussian RV. The mean signal level of $\dot{x}$ is:

$$\overline{\dot{x}}=\overline{\dot{x}_1}x_2x_3+x_1\overline{\dot{x}_2}x_3+x_1x_2\overline{\dot{x}_3}=0 \tag{7}$$

because:

$$\dot{x}_1=\dot{x}_2=\dot{x}_3=0. \tag{8}$$

The variance of $\dot{x}$ is:

$$\sigma_{\dot{x}}^2=x_2^2x_3^2\sigma_{\dot{x}_1}^2+x_1^2x_3^2\sigma_{\dot{x}_2}^2+x_1^2x_2^2\sigma_{\dot{x}_3}^2 \tag{9}$$

where:

$$\sigma_{\dot{x}_1}^2=\pi^2f_m^2\frac{\Omega_1}{m}$$

$$\sigma_{\dot{x}_2}^2=\pi^2f_m^2\frac{\Omega_2}{\kappa+1} \tag{10}$$

$$\sigma_{\dot{x}_3}^2=\pi^2f_m^2\Omega_3,$$

with f_m being maximal Doppler frequency.

After substituting, the expression for variance becomes:

$$\sigma_{\dot{x}}^2=\pi^2f_m^2\left(x_2^2x_3^2\frac{\Omega_1}{m}+x_1^2x_3^2\frac{\Omega_2}{\kappa+1}+x_1^2x_2^2\Omega_3\right)=$$

$$=\pi^2f_m^2x_2^2x_3^2\frac{\Omega_1}{m}\left(1+\frac{x^2}{x_2^4x_3^2}\frac{\Omega_2}{\Omega_1}\frac{m}{\kappa+1}+\frac{x^2}{x_2^2x_3^4}\frac{\Omega_3}{\Omega_1}m\right) \tag{11}$$

The joint probability density function of x, $\dot{x}$, x_2 and x_3 is:

$$p_{x\dot{x}x_2x_3}(x\dot{x}x_2x_3)=p_{\dot{x}}(\dot{x}/xx_2x_3)\,p_x(x/x_2x_3)\,p_{x_2}(x_2)\,p_{x_3}(x_3) \tag{12}$$

where

$$p_x(x/x_2x_3)=\left|\frac{dx_1}{dx}\right|p_{x_1}\left(\frac{x}{x_2x_3}\right) \tag{13}$$

$$\frac{dx_1}{dx}=\frac{1}{x_2x_3}. \tag{14}$$

The joint probability density function of x and $\dot{x}$ is:

$$p_{x\dot{x}}(x\dot{x})=\int_0^\infty dx_2\int_0^\infty dx_3\,p_{\dot{x}}(\dot{x}/xx_2x_3)\frac{1}{x_2x_3}p_{x_1}\left(\frac{x}{x_2x_3}\right)p_{x_2}(x_2)\,p_{x_3}(x_3) \tag{15}$$

Level crossing rate of x is [18]:

$$N_x=\int_0^\infty d\dot{x}\,\dot{x}\,p_{x\dot{x}}(x\dot{x})=$$

$$=\int_0^\infty dx_2\int_0^\infty dx_3\frac{1}{x_2x_3}p_{x_1}\left(\frac{x}{x_2x_3}\right)p_{x_2}(x_2)\,p_{x_3}(x_3)\int_0^\infty d\dot{x}\,\dot{x}\,p_{\dot{x}}(\dot{x}/xx_2x_3)=$$

$$=\int_0^\infty dx_2\int_0^\infty dx_3\frac{1}{x_2x_3}p_{x_1}\left(\frac{x}{x_2x_3}\right)p_{x_2}(x_2)\,p_{x_3}(x_3)\frac{1}{\sqrt{2\pi}}\sigma_{\dot{x}}=$$

$$= \int_0^\infty dx_2 \int_0^\infty dx_3 \frac{1}{x_2 x_3} p_{x_1}\left(\frac{x}{x_2 x_3}\right) p_{x_2}(x_2)\, p_{x_3}(x_3) \cdot$$

$$\cdot \frac{1}{\sqrt{2\pi}} \pi f_m x_2 x_3 \frac{\Omega_1^{1/2}}{m^{1/2}} \left(1 + \frac{x^2}{x_2^4 x_3^2} \frac{\Omega_2}{\Omega_1} \frac{m}{\kappa+1} + \frac{x^2}{x_2^2 x_3^4} \frac{\Omega_3}{\Omega_1} m\right)^{1/2}$$

$$= \frac{1}{\sqrt{2\pi}} \pi f_m \frac{\Omega_1^{1/2}}{m^{1/2}} \cdot \int_0^\infty dx_2 \int_0^\infty dx_3\, p_{x_1}\left(\frac{x}{x_2 x_3}\right) p_{x_2}(x_2)\, p_{x_3}(x_3) \cdot$$

$$\cdot \left(1 + \frac{x^2}{x_2^4 x_3^2} \frac{\Omega_2}{\Omega_1} \frac{m}{\kappa+1} + \frac{x^2}{x_2^2 x_3^4} \frac{\Omega_3}{\Omega_1} m\right)^{1/2} =$$

$$= \frac{1}{\sqrt{2\pi}} \pi f_m \frac{\Omega_1^{1/2}}{m^{1/2}} \cdot \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega_1}\right)^m x^{2m-1} \frac{2(\kappa+1)}{\Omega_2} \sum_{i_1=0}^{\infty} \left(\frac{\kappa(\kappa+1)}{\Omega_2}\right)^{i_1} \frac{1}{(i_1!)^2}$$

$$\frac{2}{\Omega_3} \int_0^\infty dx_2 \int_0^\infty dx_3 x_2^{-2m+1+2i_1+1} x_3^{-2m+1+1} e^{-\frac{m}{\Omega_1}\frac{x^2}{x_2^2 x_3^2} - \frac{\kappa+1}{\Omega_2} x_2^2 - \frac{1}{\Omega_3} x_3^2}$$

$$\cdot \left(1 + \frac{x^2}{x_2^4 x_3^2} \frac{\Omega_2}{\Omega_1} \frac{m}{\kappa+1} + \frac{x^2}{x_2^2 x_3^4} \frac{\Omega_3}{\Omega_1} m\right)^{1/2} \quad (16)$$

Previous two-fold integral can be solved using Laplace approximation theorem for solution the two-fold integrals [19]:

$$\int_0^\infty dx_2 \int_0^\infty dx_3 g(x_2, x_3) e^{\lambda f(x_2, x_3)} = \frac{\pi}{\lambda} \frac{g(x_{20}, x_{30})}{B(x_{20}, x_{30})} e^{\lambda f(x_{20}, x_{30})} \quad (17)$$

where B is matrix:

$$B(x_{20}, x_{30}) = \begin{vmatrix} \frac{\partial^2 f(x_{20}, x_{30})}{\partial x_{20}^2} & \frac{\partial^2 f(x_{20}, x_{30})}{\partial x_{20} \partial x_{30}} \\ \frac{\partial^2 f(x_{20}, x_{30})}{\partial x_{20} \partial x_{30}} & \frac{\partial^2 f(x_{20}, x_{30})}{\partial x_{30}^2} \end{vmatrix}. \quad (18)$$

and x_{20} and x_{30} are solution of the equations:

$$\frac{\partial f(x_{20}, x_{30})}{\partial x_{20}} = 0 \ ; \quad \frac{\partial f(x_{20}, x_{30})}{\partial x_{30}} = 0 \ . \quad (19)$$

For considered case, it is:

$$g(x_2, x_3) = x_2^{-2m+2i_1+2} x_3^{-2m+2} \left(1 + \frac{x^2}{x_2^4 x_3^2} \frac{\Omega_2}{\Omega_1} + \frac{x^2}{x_2^2 x_3^4} \frac{\Omega_3}{\Omega_1}\right)^{\frac{1}{2}} \quad (20)$$

$$f(x_2, x_3) = -\frac{m}{\Omega_1} \frac{x^2}{x_2^2 x_3^2} - \frac{(\kappa+1)}{\Omega_2} x_2^2 - \frac{1}{\Omega_3} x_3^2 \quad (21)$$

$$\frac{\partial f(x_2, x_3)}{\partial x_2} = \frac{2m}{\Omega_1} \frac{x^2}{x_2^3 x_3^2} - \frac{2(\kappa+1)}{\Omega_2} x_2 \quad (22)$$

$$\frac{\partial f(x_2, x_3)}{\partial x_3} = \frac{2m}{\Omega_1} \frac{x^2}{x_2^2 x_3^3} - \frac{2}{\Omega_3} x_3 \ . \quad (23)$$

$$\frac{2m}{\Omega_1} \frac{x^2}{x_2^3 x_3^2} - \frac{2(\kappa+1)}{\Omega_2} x_2 = 0 \quad (24)$$

$$\frac{2m}{\Omega_1} \frac{x^2}{x_2^2 x_3^3} - \frac{2}{\Omega_3} x_3 = 0 \ . \quad (25)$$

The solutions of the next two equations are x_{20} and x_{30}:

$$\frac{2m}{\Omega_1} \frac{x^2}{x_2^3 x_3^2} - \frac{2(\kappa+1)}{\Omega_2} x_2 = 0 \quad (26)$$

$$\frac{2m}{\Omega_1} \frac{x^2}{x_2^2 x_3^3} - \frac{2}{\Omega_3} x_3 = 0 \ . \quad (27)$$

They should be introduced in (17) for solving two-fold integral from (18). In this manner LCR of the product of Nakagami-*m*, Rician and Rayleigh random variables will be obtained in closed form.

III. NUMERICAL RESULTS

The level crossing rate of product of Nakagami-m random variable, Rician RV and Rayleigh RV is shown in the next few figures versus resulting signal x for different values of fading parameters and signal powers.

Dependence of the LCR, normalized by f_m, from resulting signal x, for various parameters values m and Ω_1 is presented in Fig. 1. It is possible to notice that the LCR increases for lower values of resulting signal and falls for greater values of resulting signal. All curves reach the maximum and start to decline. Smaller values of resulting signal have a greater impact to the LCR.

LCR grows for little values of Nakagami-*m* small scale fading parameter m. The impact of resulting x on the LCR is bigger for smaller magnitudes of the parameter m. The LCR is bigger for smaller value of m.

From this picture, the influence of the power Ω_1 can be observed. For low values of x, LCR increases with reduction of power Ω_1, but for bigger values of x LCR increases with growth of power Ω_1.

Fig. 2 shows the influence of the other two parameters: Rician factor κ and signal power Ω_2.

The LCR becomes bigger with enlarging of Rician factor κ. The influence of x on the LCR is greater for lower values of Rician factor κ. The impact of Nakagami-*m* fading parameter m on the LCR is higher for bigger values of Rician factor κ. From this figute one can also see that LCR is larger for greater values of power Ω_2.

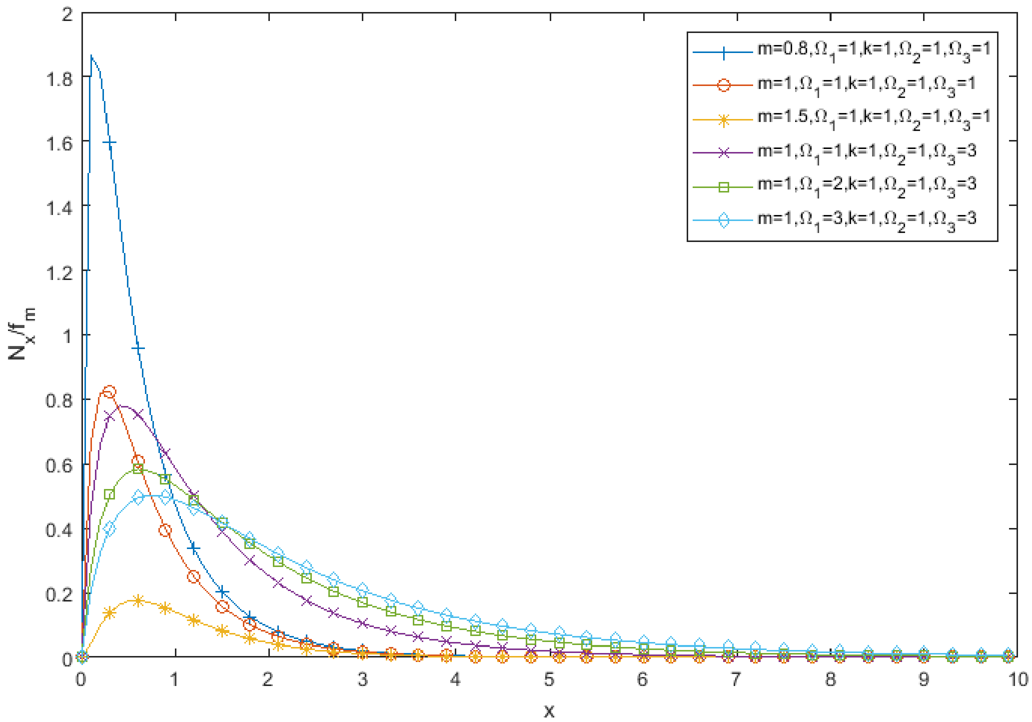

Fig. 1. LCR normalized by f_m for various values of parameters *m* and Ω_1

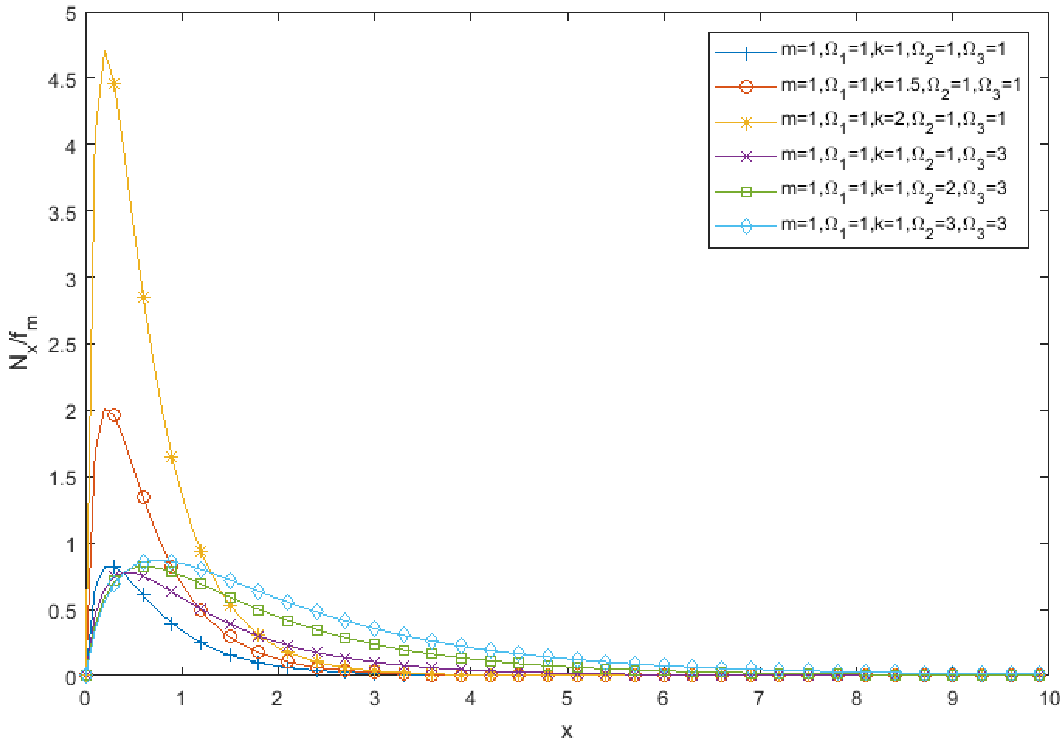

Fig. 2. LCR normalized by f_m for different parameters κ and Ω_2

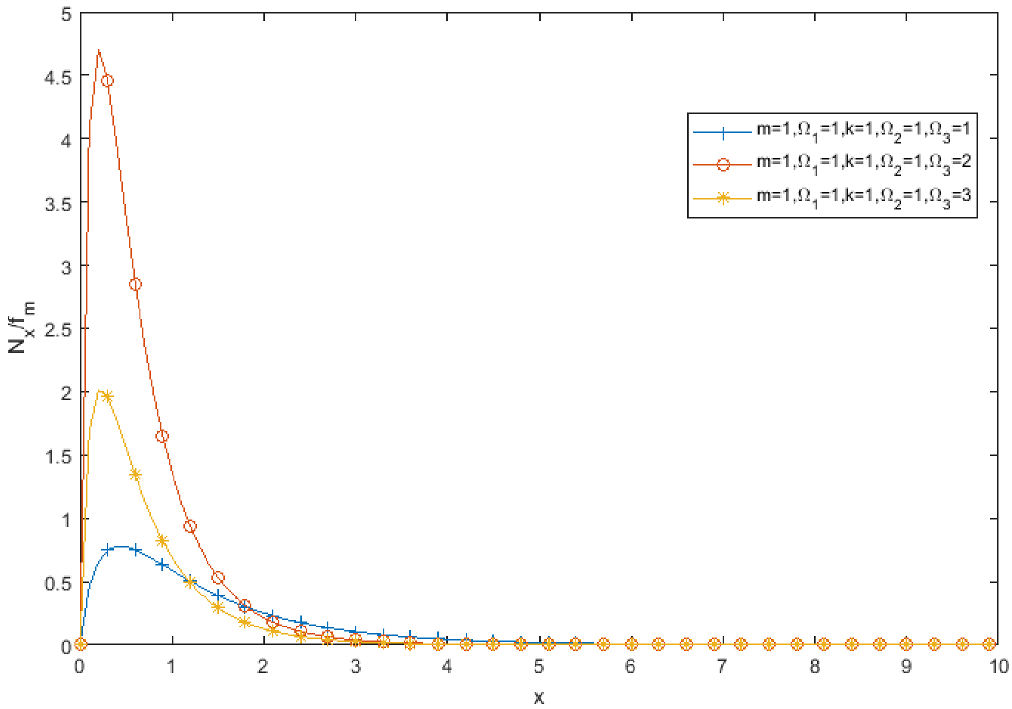

Fig. 3. LCR normalized by f_m for various values of Ω_3.

In the last figure, Fig. 3, the impact of power Ω_3 is shown. One can remark from this picture that LCR is higher for bigger values of Ω_3 and low values of x; for higher values of x, LCR is greater for smaller Ω_3. Small resulting signal x has greater impact to the LCR.

IV. Conclusion

In this article, product of Nakagami-*m* random variable, Rician and Rayleigh random variables is considered. The LCR of product of Nakagami-*m*, Rician and Rayleigh random variables is calculated. The obtained result can be used for evaluation the LCR of product of Nakagami-*m* and two Rayleigh RVs, LCR of Rician and two Rayleigh RVs and the LCR of product of three Rayleigh (3* Rayleigh) RVs. This can be achieved because Nakagami-*m* and Rician distributions are general distributions and by putting parameters *m* and κ to have a certain value, Rayleigh distribution can be obtained [2].

Also, obtained results can be used for evaluation the AFD of relay wireless communication system with three sections in the presence of Nakagami-*m* fading in the first section, Rician fading in the second section and Rayleigh fading in the third section. To the best authors' knowledge, the LCR of product of Nakagami-*m* random process, Rician random process and Rayleigh random process is not processed in open technical literature. Obtained results can be applied in performance analysis of wireless relay communication systems.

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