Mechanical Analogy of Coupled LC Resonant Circuits

#Young-Ki Cho [1], Jin-Young Choi [1], Heung Soo Kim[2] and Vakhtang Jandieri [1]
[1] School of Electronics Engineering, Kyungpook National University
[2]1370 Sangkyuk-dong, Buk-gu, Daegu, Korea, ykcho@ee.knu.ac.kr
Dept. of Telecommunication Engieering, Jeju National University, sookh@jejun.ac.kr

Abstract

Analogy is made between the coupling characteristics for the string-coupled pendulum oscillator problem and the capacitively coupled LC resonant circuit problem. This comparative study between the two problems may be helpful for undergraduate students to understand the coupling phenomena between two coupled LC resonant circuits.

Keywords : Coupled pendulum, Coupled LC circuit

1. Introduction

As it is well known, there are various coupled oscillator problems which have the same spatial symmetry and give equations of motions and normal modes with the same mathematical form as that[1] of coupled LC resonant circuits in Fig. 1. Among the various coupled oscillator problems, the most representative are the following ones, which frequently appear in the textbooks :

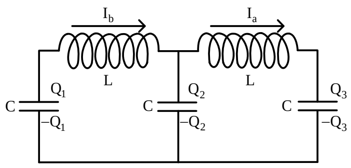

Figure 1. Two coupled LC resonant circuits

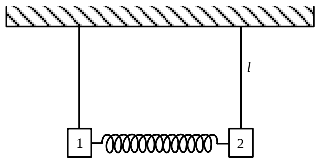

Figure 2. Two pendulums coupled by a spring

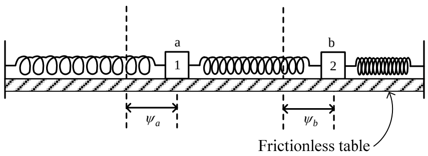

Figure 3. Longitudinal oscillator of two coupled masses

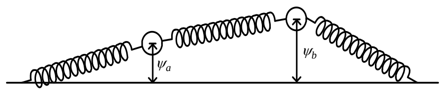

Figure 4. Transverse oscillator of two coupled masses

Fig. 2 shows the two pendulums coupled by a spring[2]. The pendulum consists of rigid rods pivoted at the top so they are oscillating without friction in the plane of the paper. The masses at the ends of the rods are coupled by a spring. Fig. 3 shows the longitudinal oscillator of two coupled masses[3]. The two masses M slide on a frictionless table. The three springs are massless and identical, each with the same spring constant. General configuration of the transverse oscillator of two coupled masses[4] is shown in Fig. 4. The oscillations are assumed to be confined to the plane of the paper. Therefore there are just two degrees of freedom in all the cases in the above 4 figures.

Coupled oscillators are examples of resonant energy exchange that is an interesting topic for many students in various majors, such as physics, chemistry, and electrical and mechanical engineering. In particular, the coupled LC resonant circuit can be introduced in the subject of circuit theory for sophomore in the engineering curricula. Furthermore, understanding the physics of the coupled LC resonant circuit is very important to understand the basic concept of the transmission line theory. However, the subject matter of the coupled oscillators is considered too advanced for freshmen and sophomores, usually because of the level of mathematics involved. Mathematical treatment of the coupled oscillator problem leads to a steady state solution of motion, which is expressed by the superposition of the normal modes.

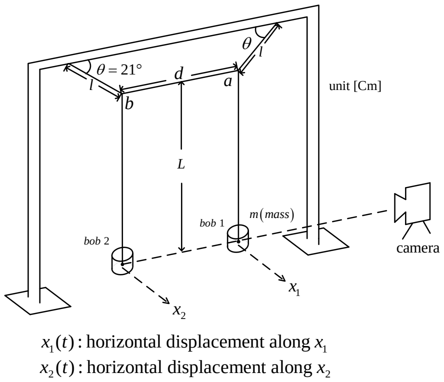

Figure 5. String-coupled pendulum oscillator and camera location for video analysis. Here $l = 27$ [Cm], $d = 35$ [Cm], L (distance between node a(b) and center of bob1(2)) $= 47.5$ [Cm] and $\theta = 21°$

In this article, we are going to consider the spring-coupled pendulum problem in Fig. 5 and compare this motion equation with that of the coupling phenomena in coupled LC resonant circuits in Fig. 1. Compared with other coupled oscillator problem, studying the present string-coupled pendulum oscillator it is easier to understand the physical meaning of two normal frequencies and how these two normal mode solutions are superposed to yield the desired steady state solution for a coupled oscillator problem.

2. Theory

First let us consider the coupling problem in Fig. 5. For linearization of the problem, the oscillation amplitudes of each bob is assumed to be so small that specification of only two horizontal displacements $x_1(t)$ and $x_2(t)$ of the right and left pendulum bobs are enough to describe the present coupled pendulum problem. The normal mode form of the equations of motion for the horizontal displacements $x_1(t)$ and $x_2(t)$ of each bob can then be constructed as follows:

$$m\frac{d^2x_1}{dt^2} + m\frac{g}{l_s}x_1 = mK(x_2 - x_1) \tag{1}$$

$$m\frac{d^2x_2}{dt^2} + m\frac{g}{l_s}x_2 = -mK(x_2 - x_1) \tag{2}$$

where the right- and left- hand pendulum mass are the same and equal to m, g is the acceleration $[9.8\mathrm{m/sec}^2]$ due to gravity, and $K = (1/2)(\omega_a{}^2 - \omega_s{}^2) = (1/2)\left(g/l_a - g/l_s\right)$. Here $\omega_a = \sqrt{g/l_a}$ and $\omega_s = \sqrt{g/l_s}$ mean natural frequencies for anti-symmetrical and symmetrical modes respectively, $l_a\left(= L + (d/2 \cdot l\sin\theta)/(d/2 + l\cos\theta)\ [\mathrm{Cm}]\right)$ and $l_s\left(= L + l\sin\theta\ [\mathrm{Cm}]\right)$ mean the effective length corresponding to ω_a and ω_s respectively. Adding eqn.(1) and eqn.(2) gives

$$\frac{d^2X}{dt^2} + \omega_s^2 X = 0 \tag{3}$$

and subtracting eqn.(2) from eqn.(1) gives

$$\frac{d^2Y}{dt^2} + \omega_a^2 Y = 0 \tag{4}$$

where $X = x_1 + x_2$, $Y = x_1 - x_2$, and $\omega_a^2 = \omega_s^2 + 2K = g / l_a$. These expressions for the effective lengths, l_s and l_a of the pendulum for both symmetrical and anti-symmetrical modes are the same as those in previous work[4]. This validates the normal mode expressions in eqn.(1) and eqn.(2).

It is well known that all the motions of the horizontal displacement of bobs can be represented as linear combinations (superposition) of these normal modes. Thus, we can expect the motions of the horizontal displacements, $x_1(t)$ and $x_2(t)$ of bob 1 and bob 2 to be represented by the sum of the two following cosine functions:

$$x_1 = A_s \cos(\omega_s t + \phi_s) + A_a \cos(\omega_a t + \phi_a) \tag{5}$$

$$x_2 = A_s \cos(\omega_s t + \phi_s) - A_a \cos(\omega_a t + \phi_a) \tag{6}$$

where the subscripts s and a refer to the symmetric and anti-symmetric modes respectively. As it is seen from eqn.(5) and eqn.(6), combining the two normal modes, both at an equal maximum amplitude $A_s = A_a = A$, displaces bob 1 by $2A$ and leaves bob 2 in equilibrium. We can now identify the four constants in eqn.(5) and eqn.(6): $A_s = A_a = A = 32\text{mm}$, $\phi_a = \phi_s = 0$ (both normal modes are at their maximum amplitude at $t = 0$). When using the above normal mode angular frequencies, ω_s and ω_a along with the above four constants, we can then predict that

$$x_1(t) = 2A\cos(\frac{\omega_s + \omega_a}{2}t)\cos(\frac{\omega_s - \omega_a}{2}t) \tag{7}$$

$$x_2(t) = -2A\sin(\frac{\omega_s + \omega_a}{2}t)\sin(\frac{\omega_s - \omega_a}{2}t) \tag{8}$$

In case of the two coupled LC circuit in Fig. 1, equations for the two branch currents, I_a and I_b, are given by the following expressions:

$$\frac{d^2 I_a}{dt^2} + \omega_s{}^2 I_a = K\left(I_b - I_a\right) \tag{9}$$

$$\frac{d^2 I_b}{dt^2} + \omega_s{}^2 I_b = -K\left(I_b - I_a\right) \tag{10}$$

Adding eqn.(9) and eqn.(10) gives

$$\frac{d^2 \tilde{X}}{dt^2} + \omega_s{}^2 \tilde{X} = 0 \tag{11}$$

and subtracting eqn.(10) from eqn.(9) gives

$$\frac{d^2 \tilde{Y}}{dt^2} + \omega_a{}^2 \tilde{Y} = 0 \tag{12}$$

where $\tilde{X} = I_a + I_b$, $\tilde{Y} = I_a - I_b$, $\omega_s{}^2 = \frac{1}{LC}$, and $\omega_a{}^2 = \omega_s{}^2 + 2K = \frac{3}{LC}$. For all the physical phenomena having two degrees of freedom, the equation of motion is expressed as a superposition of two harmonics oscillations having different angular frequencies ω_s and ω_a as defined above.

The mechanically coupled pendulum problem in Fig. 5 is analogous to the problem of the coupled LC circuit in Fig. 1, where the relation between two systems are :

$$I_a(I_b) \leftrightarrow x_1(x_2) \tag{13}$$

$$\omega_s{}^2\left(= g / l_s\right) \leftrightarrow \omega_s{}^2\left(=1/(LC)\right) \tag{14}$$

$$\omega_a{}^2\left(=g/l_a\right) \leftrightarrow \omega_a{}^2\left(=3/(LC)\right) \tag{15}$$

$$K=\frac{1}{2}\left(\omega_a{}^2-\omega_s{}^2\right) \leftrightarrow K=\frac{1}{2}\left(\omega_a{}^2-\omega_s{}^2\right) \tag{16}$$

$$=\frac{1}{2}\left(\frac{g}{l_a}-\frac{g}{l_s}\right) \leftrightarrow =\frac{1}{2}\left(\frac{3}{LC}-\frac{1}{LC}\right)=\frac{1}{LC} \tag{17}$$

3. Validation

The solution for the horizontal displacements, x_1 and x_2, or the branch current I_1 and I_2 is given in the form of eqns.(7) and (8). If we plot the behavior of the individual bob by showing x_1 and x_2 change with time, we see that after releasing the right hand pendulum bob, $x_1(t)$ follows a cosinusoidal behavior at a frequency which is the average of the two normal mode frequencies. Its amplitude varies cosinusoidally with a low frequency, which is half the difference between the normal mode frequencies. On the other hand, $x_2(t)$, which started at zero, vibrates sinusoidally with the average frequency but its amplitude builds up to $2A=64[\mathrm{mm}]$ and then decays sinusoidally at the low frequency of half the difference between the normal mode frequencies. Fig. 6 illustrates horizontal displacements $x_1(t)$ and $x_2(t)$ for the case $l=27[\mathrm{cm}]$, $d=35[\mathrm{cm}]$, $L=47.5[\mathrm{cm}]$ and $\theta=21°$.

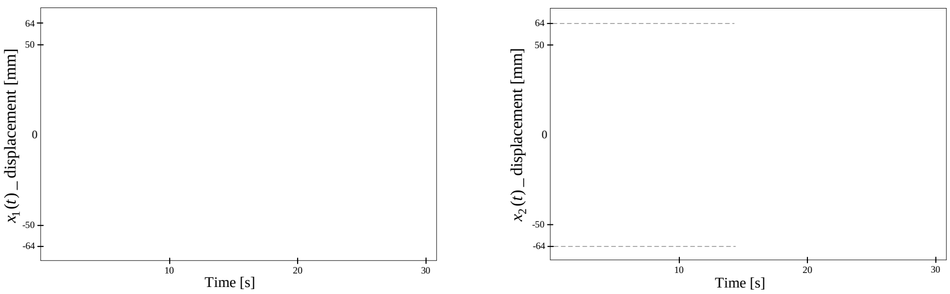

Figure 6. Horizontal displacements $x_1(t)$ and $x_2(t)$ of the pendulum bob 1 and 2

This procedure for the mechanical problem in Fig. 1 can be applied to various problems by direct analogy.

4. Conclusion

As a mechanically analogous problem to capacitively coupled LC resonant circuit, a string coupled pendulum oscillator problem was considered. The solution for the string coupled pendulum problem was compared with that for the coupled LC resonant circuit problem. This study may be helpful for students to understand the coupling phenomena in the coupled LC resonant circuit.

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