INVESTIGATION OF NUMERICAL STABILITY OF 2D FE/FDTD HYBRID ALGORITHM FOR DIFFERENT HYBRIDIZATION SCHEMES

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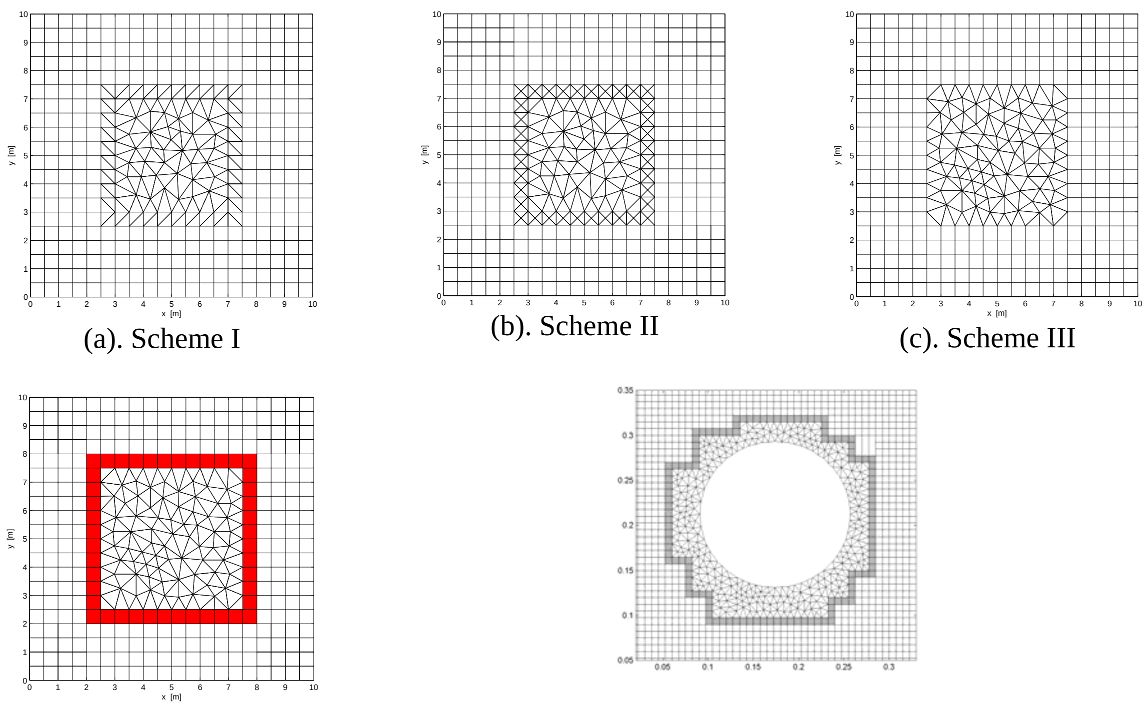
1. Introduction

The Finite-Difference Time-Domain (FDTD) Method is one of the efficient time domain methods widely used in studying the transient and wide-band phenomena in various electromagnetic and microwave systems. Stair-case approximation of the boundary of the modeled geometry is a major source of errors in the FDTD algorithm. To overcome stair-casing errors, the FDTD algorithm has been hybridized with the finite element method (FEM) in the time domain [1]. This hybrid algorithm facilitates the accurate modeling of geometries by meshing the region in the vicinity of the geometry using unstructured grids conforming to the geometry, while other parts of the physical domain is modeled using traditional FDTD method with Cartesian grids. In this communication, we compare the numerical stability of the time-marching algorithm for four different hybridization schemes. An eigenvalue analysis of the global iteration matrix, representing the time marching hybrid algorithm is performed to compare and analyze the numerical stability of the different hybridization schemes. Further numerical experiments are performed to verify the stability analysis.

2. Hybrid Algorithm Formulation

The original FE-FDTD method proposed in [1] splits the physical space into two overlapping domains viz., the Finite Difference, Ω_{FD} and Finite element, Ω_{FE} regions. The overlapping region is



(a). Scheme I (b). Scheme II (c). Scheme III

(d). Scheme IV (e). PEC Cylinder model

Fig.1 Hybrid Mesh for the different hybridization strategies.

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one FDTD cell thick. Traditional leapfrog scheme on a staggered grid is used for the unknowns in Ω_{FD}. Triangular elements with unconditionally stable Newmark -β scheme is used for temporal discretization for the unknowns in Ω_{FE}. Scheme I is that used in [1],[2]. Scheme II is similar to the one followed in [2], where the one FDTD cell thick overlapping region between the two domains is triangulated into four triangular elements, instead of two as used in [2]. Scheme III follows the strategy proposed in [3]. In [4], Scheme II was shown to introduce less unphysical reflections in the numerical solution and hence has better accuracy compared to Scheme I and III. In the current proposed hybridization Scheme IV, Ω_{FE} is discretized such that the overlapping region is modeled using rectangular edge elements and the rest using triangular finite elements. The hybrid mesh for the four different schemes is shown in Fig .1. In Fig. 4(d) and (e), the shaded cells are the rectangular elements in Ω_{FE}.

3. Stability analysis

Temporal instabilities often arise in the FE-FDTD hybrid algorithm [2,3]. In [5] filtering techniques are proposed to control the stability of the algorithm. In [3], Backward Difference Formula (BDF-2) method is used for temporal discretization to improve the stability. Numerical stability of a time-marching algorithm represented by

$$\mathbf{v}^{n+1} = \mathbf{G}(\Delta t, \Delta h)\mathbf{v}^{n} \tag{1}$$

where $\mathbf{v}^n$ is the unknown at time $n\Delta t$ (1) can be investigated by analyzing the eigenvalues of the global iteration matrix, $\mathbf{G}$. The necessary condition for stability is $\rho(\mathbf{G}) \le 1$ where $\rho(\mathbf{G})$ is the spectral radius of the matrix $\mathbf{G}$ [6]. To obtain $\mathbf{G}$, the update equations for the unknowns in Ω_{FD} and Ω_{FE} have to be combined. In what follows, the notation used are: $\mathbf{e}_{\mathrm{FD}}$ is the electric field unknowns in Ω_{FD}, $\mathbf{e}_{\overline{\mathrm{FE}}}$ is the electric field unknowns on the boundary of Ω_{FE}, $\mathbf{e}_{\overline{\mathrm{FD}}}$ are the field unknowns in the overlapping region that conform to the grid in Ω_{FD} and finally, $\mathbf{e}_{\mathrm{FE}}$ are the unknowns in Ω_{FE}. Consider the FDTD update equation for unknowns in Ω_{FD}. By eliminating the magnetic field unknowns, we obtain

$$\begin{bmatrix} \mathbf{e}_{\mathrm{FD}} \\ \mathbf{e}_{\overline{\mathrm{FE}}} \\ \mathbf{e}_{\overline{\mathrm{FD}}} \end{bmatrix}^{n+1} = \begin{bmatrix} 2\mathbf{I}+\mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & 2\mathbf{I}+\mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & 2\mathbf{I}+\mathbf{A}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{e}_{\mathrm{FD}} \\ \mathbf{e}_{\overline{\mathrm{FE}}} \\ \mathbf{e}_{\overline{\mathrm{FD}}} \end{bmatrix}^{n} - \begin{bmatrix} \mathbf{e}_{\mathrm{FD}} \\ \mathbf{e}_{\overline{\mathrm{FE}}} \\ \mathbf{e}_{\overline{\mathrm{FD}}} \end{bmatrix}^{n-1} \tag{2}$$

The implicit update equation for the unknowns in Ω_{FE} is written as

$$\begin{bmatrix} \mathbf{M}_{33} & \mathbf{M}_{23}^{t} & \mathbf{M}_{13}^{t} \\ \mathbf{M}_{23} & \mathbf{M}_{22} & \mathbf{M}_{12}^{t} \\ \mathbf{M}_{13} & \mathbf{M}_{12} & \mathbf{M}_{11} \end{bmatrix} \begin{bmatrix} \mathbf{e}_{\overline{\mathrm{FE}}} \\ \mathbf{e}_{\overline{\mathrm{FD}}} \\ \mathbf{e}_{\mathrm{FE}} \end{bmatrix}^{n+1} = \begin{bmatrix} \mathbf{N}_{33} & \mathbf{N}_{23}^{t} & \mathbf{N}_{13}^{t} \\ \mathbf{N}_{23} & \mathbf{N}_{22} & \mathbf{N}_{12}^{t} \\ \mathbf{N}_{13} & \mathbf{N}_{12} & \mathbf{N}_{11} \end{bmatrix} \begin{bmatrix} \mathbf{e}_{\overline{\mathrm{FE}}} \\ \mathbf{e}_{\overline{\mathrm{FD}}} \\ \mathbf{e}_{\mathrm{FE}} \end{bmatrix}^{n} - \begin{bmatrix} \mathbf{M}_{33} & \mathbf{M}_{23}^{t} & \mathbf{M}_{13}^{t} \\ \mathbf{M}_{23} & \mathbf{M}_{22} & \mathbf{M}_{12}^{t} \\ \mathbf{M}_{13} & \mathbf{M}_{12} & \mathbf{M}_{11} \end{bmatrix} \begin{bmatrix} \mathbf{e}_{\overline{\mathrm{FE}}} \\ \mathbf{e}_{\overline{\mathrm{FD}}} \\ \mathbf{e}_{\mathrm{FE}} \end{bmatrix}^{n-1} \tag{3}$$

The matrices, $\mathbf{M}$ and $\mathbf{N}$ are given as

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{33} & \mathbf{M}_{23}^{t} & \mathbf{M}_{13}^{t} \\ \mathbf{M}_{23} & \mathbf{M}_{22} & \mathbf{M}_{12}^{t} \\ \mathbf{M}_{13} & \mathbf{M}_{12} & \mathbf{M}_{11} \end{bmatrix} = \mathbf{T} + \beta c^2 \Delta t^2 \mathbf{S} \text{ and } \mathbf{N} = \begin{bmatrix} \mathbf{N}_{33} & \mathbf{N}_{23}^{t} & \mathbf{N}_{13}^{t} \\ \mathbf{N}_{23} & \mathbf{N}_{22} & \mathbf{N}_{12}^{t} \\ \mathbf{N}_{13} & \mathbf{N}_{12} & \mathbf{N}_{11} \end{bmatrix} = 2\mathbf{T} - (1-2\beta)c^2\Delta t^2 \mathbf{S}$$

where $\mathbf{T}$ and $\mathbf{S}$ are the traditional finite element mass and stiffness matrices, respectively. β is the parameter in Newmark- β schemes which is unconditionally stable for $\beta \ge 0.25$. For the FE/FDTD hybrid algorithm, (2) and (3) are combined to obtain a single 2-step electric field update equation given as

$$\mathbf{Q}_1 \mathbf{e}^{n+1} = \mathbf{Q}_0 \mathbf{e}^{n} - \mathbf{Q}_1 \mathbf{e}^{n-1} \tag{4}$$

where the matrices $\mathbf{Q}_1$ and $\mathbf{Q}_0$ and the vector $\mathbf{e}$ are defined as

$$\mathbf{Q}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{23} & \mathbf{M}_{22} & \mathbf{M}_{12}^t \\ \mathbf{0} & \mathbf{M}_{13} & \mathbf{M}_{12} & \mathbf{M}_{11} \end{bmatrix}; \quad \mathbf{Q}_0 = \begin{bmatrix} 2\mathbf{I}+\mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} & \mathbf{0} \\ \mathbf{A}_{21} & 2\mathbf{I}+\mathbf{A}_{22} & \mathbf{A}_{23} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_{23} & \mathbf{N}_{22} & \mathbf{N}_{12}^t \\ \mathbf{0} & \mathbf{N}_{13} & \mathbf{N}_{12} & \mathbf{N}_{11} \end{bmatrix} \text{ and } \mathbf{e} = \begin{bmatrix} \mathbf{e}_{\mathrm{FD}} \\ \mathbf{e}_{\overline{\mathrm{FE}}} \\ \mathbf{e}_{\overline{\mathrm{FD}}} \\ \mathbf{e}_{\mathrm{FE}} \end{bmatrix}$$

Eq. (4) can be written in the form of (1) such that

$$\mathbf{v}^n = \begin{bmatrix} \mathbf{e}^n \\ \mathbf{e}^{n-1} \end{bmatrix} \text{ and } \mathbf{G} = \begin{bmatrix} \mathbf{Q}_1^{-1}\mathbf{Q}_0 & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \tag{5}$$

Using (5) it can be shown that the eigenvalues of $\mathbf{G}$, $\lambda_{\mathbf{G}}$ are related to the eigenvalues of $\mathbf{Q}_1^{-1}\mathbf{Q}_0$, $\lambda_{\mathbf{Q}}$ as

$$\lambda_{\mathbf{G}} = \frac{\lambda_{\mathbf{Q}}}{2} \pm \sqrt{\left(\frac{\lambda_{\mathbf{Q}}}{2}\right)^2 - 1} \tag{6}$$

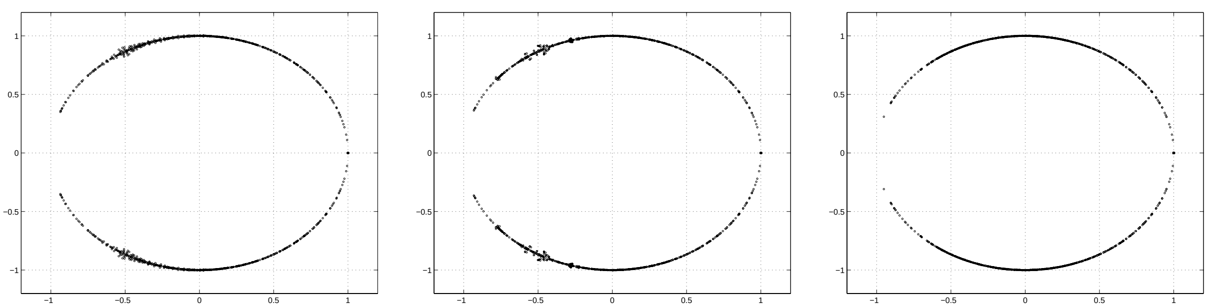
Upon computing $\mathbf{Q}_0$ and $\mathbf{Q}_1$ for the given hybrid mesh, the eigenvalues of the iteration matrix $\lambda_{\mathbf{G}}$ and hence $\rho(\mathbf{G})$ can be computed. $\rho(\mathbf{G}) \le 1$ is the necessary condition for stability (though not sufficient, since $\mathbf{G}$ is not a normal matrix [6]).

4. Numerical Experiments

The computation of the $\mathbf{Q}_0$ and $\mathbf{Q}_1$ is straight forward in Schemes I, II and IV since the unknowns $\mathbf{e}_{\overline{\mathrm{FE}}}$ and $\mathbf{e}_{\overline{\mathrm{FD}}}$ in (1) and (2) conform to each other, respectively. However, this is not the case in scheme III, which is also known to lead to instabilities [3]. An eigenvalue analysis is not performed on this scheme. It is worth noting that in Scheme II, $\mathbf{M}_{23} = \mathbf{N}_{23} = \mathbf{0}$ and in Scheme IV, $\mathbf{M}_{13} = \mathbf{N}_{13} = \mathbf{0}$. This is because of the nature of finite elements used in discretizing the overlapping region. The eigenvalues of the iteration matrix in schemes I, II and IV for mesh shown in Fig .1(a), (b) and (d), are displayed in Fig. 2. The space step is Δh=0.5m and time step is determined such that $c^2\Delta t^2 = 0.5\Delta h^2$. In Table 1, the spectral radius and the percentage of eigenvalues whose modulus is greater than 1 are shown. It is observed that $\rho(\mathbf{G})$ in Scheme IV is closer to 1, and hence expected to have better stability. Also, of the three schemes it has the least number of eigenvalues with modulus greater than 1. It is concluded that though all the schemes considered are unstable, Scheme IV is expected to have better stability. The four hybridization schemes are used to compute the scattering

Table 1. Eigenvalue statictis of iteration matrix for different schemes

	Scheme I	Scheme II	Scheme IV
$\rho(\mathbf{G})$	1.03769	1.04277	1.00269
% of $\lvert\lambda_{\mathbf{G}}\rvert > 1$	13.39	10.76	3.56



(a) Scheme I (b) Scheme II (C)Scheme IV

Fig.2 Eigenvalues of the iteration matrix of the different schemes

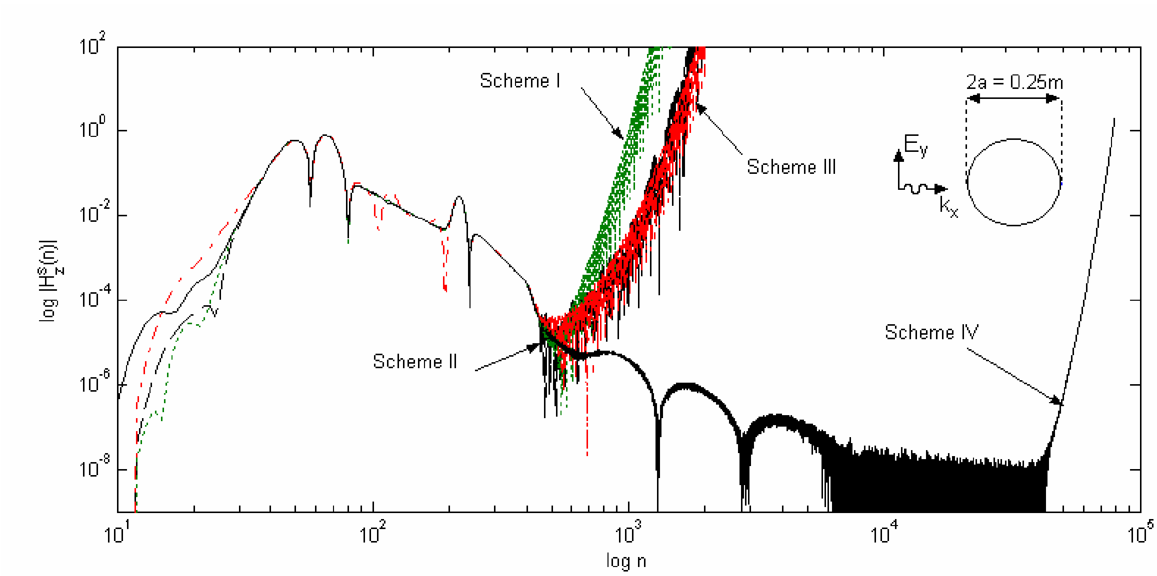


Fig. 3. Backscattered $H_z(t)$ component by a PEC cylinder using the different schemes

from a PEC cylinder. A differentiated Gaussian pulse with spectral content in the band 0.8GHz - 2 GHz is the incident along the $\hat{x}$ direction. The backscattered time domain Hz component is shown in Fig. 3. It is clearly observed that instabilities arise within 1000 time steps for Schemes I, II and III while for scheme IV it starts to appear around 60,000 time steps. Thus, Scheme IV has better numerical stability as assessed in the eigenvalue analysis in the previous section. Since the number of time steps reflects the resolution in the frequency domain, Scheme IV should be used to analyze problems with high quality factor (which demands high frequency resolution, and hence more time steps in the time domain solution).

5. Conclusion

A framework for the analysis of the numerical stability of hybrid FE/FDTD algorithm is developed. The eigenvalue analysis of the resulting iteration matrix for different hybridization schemes reveals that Scheme IV has better stability properties than others. This was validated by the numerical experiments using the various hybridization strategies. The extension of the framework to 3D is straight forward and is currently under investigation.

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