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JOINT TWO DIMENSIONAL ANGLE AND DELAY ESTIMATION USING RECTANGULAR PLANAR ARRAY

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1. Introduction

In the multipath communication scenario, it is needed to estimate the two dimensional angles and relative time delays of each multi-path ray, such as source localization, the estimation of a parametric propagation channel, to assist equalization and directive transmission in the downlink. Estimation of two dimensional angles and delays also has applications in radar, and seismic exploration.

One aspect of the multipath estimation problem which has received little attention so far is the estimation of three-dimensional parameter. Obviously, we can obtain 2-D angles and delays of the incoming signals in two vertically located ULA using the methods proposed in [1-4]. While there are two disadvantages with that, on one hand, the pairing of the 2-D angles and delays can't be automatically determined; on the other hand, if two or more rays have two close parameters simultaneously, the data covariance matrix turns ill-conditioned, and those algorithms may not work properly in the methods suggested in [1-3]; TST-MUSIC proposed in [4] can only solve the problem in the case of only one close parameter. In light of this setbacks, we present two low complexity, yet hight accuracy algorithms, modified TST-MUSIC (MTST-MUSIC) and 3D JADE-ESPRIT to estimate two dimensional angles and delays in rectangular planar array (RPA). The 3D JADE-ESPRIT algorithm uses a 3D ESPRIT-like shift-invariance technique to separate and estimate the 2-D DOAs and delays, the basic idea of MTST-MUSIC is to group and isolate the signal of each incoming ray using the space-time characteristics of the multiray wireless channel.

2. Data Model

We consider a RPA shown in Fig.1. The array consists of $D \times D$ elements. Assume that Q narrow band plane waves impinge on the RPA of the D^2 sensors from angular directions (γ_i, θ_i)(or (α_i, β_i)), where $\cos\alpha_i = \cos\gamma_i \cdot \cos\theta_i, \cos\beta_i = \cos(\pi/2 - \theta_i) \cdot \cos\gamma_i$ .We assume that the sensor located at the origin is a_{11} , the nth sensor along the x-aixs can be expressed as a_{n1} and the mth sensor along the y-aixs can be expressed as a_{1m} . The output of the a_{nm} sensor is represented by

$$x_{nm}(t) = \sum_{i=1}^{Q} s_i(t) e^{-j\frac{2\pi}{\lambda}(n-1)d\cos\alpha_i} e^{-j\frac{2\pi}{\lambda}(m-1)d\cos\beta_i} + n_{nm}(t) \tag{1}$$

where, $s_i(t)$ is the ith emitter signal and $n_{nm}(t)$ denotes the additive noise of nmth sensor, d denotes inter-element distance. The sensor outputs are collected in the complex matrix , to form the array output

$$\mathbf{X}(t) = \mathbf{A}(\alpha) diag[\mathbf{S}(t)] \mathbf{A}^T(\beta) + \mathbf{N}(t) \tag{2}$$

where

$$\mathbf{X}(t) = [\mathbf{x}_{n1}(t), \mathbf{x}_{n2}(t), \cdots, \mathbf{x}_{nD}(t)] \tag{3}$$

denotes the output of the RPA, $D \times D-$dimensional matrix. $\mathbf{x}_{ni}(t)$ is a $D \times 1$ vector such that $\mathbf{x}_{ni}(t) = [x_{1i}(t), \cdots, x_{Di}(t)]^T$. $\mathbf{S}(t)$ is a $Q \times 1$ vector given by

$$\mathbf{S}(t) = [s_1(t), \cdots, s_Q(t)]^T \tag{4}$$

$\mathbf{A}(\alpha)$ and $\mathbf{A}(\beta)$ denotes the $D \times Q$-dimensional array response matrix respectively such that

$$\mathbf{A}(\alpha) = [\mathbf{a}(\alpha_1), \cdots, \mathbf{a}(\alpha_Q)], \quad \mathbf{A}(\beta) = [\mathbf{a}(\beta_1), \cdots, \mathbf{a}(\beta_Q)] \tag{5}$$

with $\mathbf{a}(\alpha)$ and $\mathbf{a}(\beta)$ being two $D \times 1$ complex vector characterized by an unknown parameter α_k and β_k associated with the kth signal ($k = 1, \cdots, Q$), it is given by

$$\mathbf{a}(\alpha_i) = [1, e^{-j\frac{2\pi}{\lambda}d\cos\alpha_i}, \cdots, e^{-j\frac{2\pi}{\lambda}(D-1)d\cos\alpha_i}]^T, \quad \mathbf{a}(\beta_i) = [1, e^{-j\frac{2\pi}{\lambda}d\cos\beta_i}, \cdots, e^{-j\frac{2\pi}{\lambda}(D-1)d\cos\beta_i}]^T \tag{6}$$

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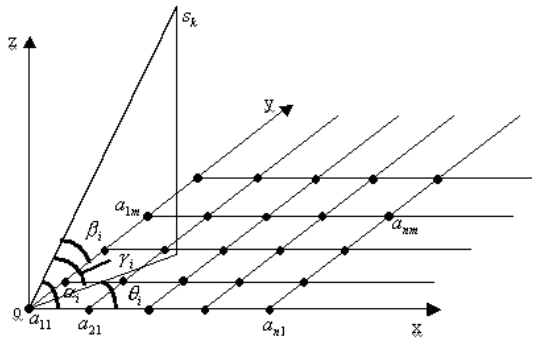

Fig. 1. Rectangular planar array geometry.

where α and β are the angle between the impinging waves and x-axis (AWX), y-axis (AWY) respectively. $\mathbf{N}(t)$ is the $D \times D$ complex noise matrix given by

$$\mathbf{N}(t) = [\mathbf{N}_{n1}(t), \mathbf{N}_{n2}(t), \cdots, \mathbf{N}_{nD}(t)] \tag{7}$$

where $\mathbf{N}_{ni}(t) = [n_{1i}(t), n_{2i}(t), \cdots, n_{Di}(t)]^T$.

From (2), it can be obtained

$$\mathbf{X}_{vec}(t) := \text{vec}(\mathbf{X}(t)) = (\mathbf{A}(\beta) \circ \mathbf{A}(\alpha))\mathbf{S}(t) + \text{vec}(\mathbf{N}(t)) \tag{8}$$

We have used the general relation vec($\mathbf{A} \quad diag[\mathbf{b}] \quad \mathbf{C}$) $= (\mathbf{C}^T \circ \mathbf{A})$b as it applies to (2). In this paper, we assume a TDMA wireless system with a known sequence, such as GSM, is our target application system. In light of [4] we have

$$\mathbf{X}_t^{(n)} = \mathbf{A}(\beta) \circ \mathbf{A}(\alpha)\mathbf{B}^{(n)}\mathbf{G}(\tau)^T + \mathbf{N} \tag{9}$$

where $\mathbf{X}_t^{(n)}$ is the signal received during the nth time burst under a rate T/P, given by

$$\mathbf{X}_t^{(n)} = [\mathbf{x}_{vec}^{(n)}(t_0), \mathbf{x}_{vec}^{(n)}(t_0 - \frac{T}{P}), \cdots \mathbf{x}_{vec}^{(n)}(t_0 - (N - \frac{1}{P})T)] \tag{10}$$

$\mathbf{G} = [\mathbf{g}(\tau_1) \cdots \mathbf{g}(\tau_Q)]$ is a matrix of dimension $Q \times NP$, with $\mathbf{g}(\tau_i)$ is the convolution between the training sequence and the time-shifted pulse-shaping function. $\mathbf{B}^{(n)} = diag\{\beta_1^{(n)}, \cdots, \beta_Q^{(n)}\}$ with $\beta_i^{(n)}$ being the complex fading amplitude of the ith ray during the nth burst. We define the $\beta(n) := [\beta_1^{(n)}, \cdots, \beta_Q^{(n)}]^T$. It is assumed that $\|\mathbf{a}(\alpha_i)\| = \|\mathbf{a}(\beta_i)\| = \|\mathbf{g}(\tau_i)\| = 1$ for all α, β and τ, by adjusting power between $\mathbf{A}(\alpha)$, $\mathbf{A}(\beta)$, $\mathbf{G}(\tau)$ and $\mathbf{B}^{(n)}$ respectively. We suppose that the transmitter signal is a complex Gaussian random process and thus have

$$\varepsilon\{\mathbf{B} \cdot \mathbf{B}^H\} = \varepsilon\{\beta(n) \cdot \beta(n)^H\} = diag(\delta_1^2, \cdots, \delta_Q^2) =: P \tag{11}$$

Where superscript$(\cdot)^T$, $(\cdot)^*$, and $(\cdot)^H$ denotes matrix transpose, complex conjugate and Hermitian transpose respectively. δ_i^2 is average signal power of ray i, $\varepsilon(\cdot)$ is statistical average. $\otimes$ is Kronecker product, $\circ$ is Khatri-Rao product [5], which is a column-wise Kronecker produt: $\mathbf{A} \circ \mathbf{B} = [\mathbf{a}_1 \otimes \mathbf{b}_1 \quad \mathbf{a}_2 \otimes \mathbf{b}_2 \quad \cdots]$. $\dagger$ is matrix pseudo-inverse, $\mathbf{I}_m$ is $m \times m$ identity matrix.

3. Proposed Algorithm

Note that (9) is different from the model in the 1-D DOA and delay estimation for different parameter dimension. However we can modify the methods suggested in [1-4] to have joint 2-D DOA and delay estimation.

A. 3D JADE-ESPRIT Algorithm

In the frequency domain, if we carry out an N_t-point discrete Fourier transform (DFT) on the rows of (9), the resulting frequency domain representation of $\mathbf{X}_t^{(n)}$ is

$$\mathbf{X}_f^{(n)} = \mathbf{A}(\beta) \circ \mathbf{A}(\alpha)\mathbf{B}^{(n)}\mathbf{V}(\tau)^T \cdot diag\{\tilde{\mathbf{g}}\} \tag{12}$$

where $\mathbf{V}(\tau) =: [\mathbf{v}(\tau_1), \cdots, \mathbf{v}(\tau_Q)]$ in which $\mathbf{v}(\tau_k) = [1, \upsilon_k, \cdots, \upsilon_k^{N_t-1}]^T$ with $\upsilon_k = e^{-j\frac{2\pi}{N_t}\tau_k}$; $\tilde{\mathbf{g}} = [g_0(0), \cdots, g_0(N_t - 1)]^T$, with $g_0(k)$ denoting the kth element of the DFT of $\mathbf{S}_t^T\mathbf{g}(0)$. N_t is Nyquist rate. (12) can be shown

$$\mathbf{H}^{(n)} = \mathbf{X}_f^{(n)} \cdot diag\{\tilde{\mathbf{g}}\}^{-1} = \mathbf{A}(\beta) \circ \mathbf{A}(\alpha)\mathbf{B}^{(n)}\mathbf{V}(\tau)^T \tag{13}$$

It is obtained that $\mathbf{vec}(\mathbf{H}^{(n)}) = \mathbf{U}(\tau, \beta, \alpha)\beta(n)$, where $\mathbf{U}(\tau, \beta, \alpha) = \mathbf{V}(\tau) \circ \mathbf{A}(\beta) \circ \mathbf{A}(\alpha)$, we have used the general relation $\mathbf{A} \circ (\mathbf{B} \circ \mathbf{C}) = \mathbf{A} \circ \mathbf{B} \circ \mathbf{C}$. Note that $\mathbf{V}(\tau)$ also possesses the rotational invariance structure.

Obviously, $\mathbf{U}(\tau,\beta,\alpha)$ has a triple Vandermonde structure, as proposed in [1], $\mathbf{E}$ containing a basis of the column span of $\mathbf{U}$ can be estimated by taking the left singular vectors corresponding to the largest $\mathbf{Q}$ singular values of $\mathbf{vec}(\mathbf{H}^{(n)})$. Thus we have $\mathbf{E}=\mathbf{U}\mathbf{T}$, where $\mathbf{T}$ is a square invertible $\mathbf{Q}\times\mathbf{Q}$ matrix. Define the following selection matrices:

$$\mathbf{J}_\alpha=\mathbf{I}_{N_t}\otimes\mathbf{I}_D\otimes[\mathbf{I}_{D-1}0_1],\quad \mathbf{J}_\beta=\mathbf{I}_{N_t}\otimes[\mathbf{I}_{D-1}0_1]\otimes\mathbf{I}_D,\quad \mathbf{J}_\tau=[\mathbf{I}_{N_t-1}0_1]\otimes\mathbf{I}_{D\times D} \tag{14}$$

$$\mathbf{J}'_\alpha=\mathbf{I}_{N_t}\otimes\mathbf{I}_D\otimes[0_1\mathbf{I}_{D-1}],\quad \mathbf{J}'_\beta=\mathbf{I}_{N_t}\otimes[0_1\mathbf{I}_{D-1}]\otimes\mathbf{I}_D,\quad \mathbf{J}'_\tau=[0_1\mathbf{I}_{N_t-1}]\otimes\mathbf{I}_{D\times D} \tag{15}$$

and we define $\mathbf{F}_\alpha=\mathbf{J}_\alpha\mathbf{U}$, and similarly for $\mathbf{F}'_\alpha,\mathbf{F}_\beta,\mathbf{F}'_\beta,\mathbf{F}_\tau,\mathbf{F}'_\tau$, thus we can obtain $\mathbf{F}'_\alpha=\mathbf{F}_\alpha\Psi,\mathbf{F}'_\beta=\mathbf{F}_\beta\Phi,\mathbf{F}'_\tau=\mathbf{F}_\tau\Upsilon$, where Ψ,Φ,Υ is shift invariance of $\mathbf{A}(\alpha),\mathbf{A}(\beta),\mathbf{G}(\tau)$ respectively. Then let

$$\mathbf{E}_\alpha:=\mathbf{J}_\alpha\mathbf{E},\quad \mathbf{E}_\beta:=\mathbf{J}_\beta\mathbf{E},\quad \mathbf{E}_\tau:=\mathbf{J}_\tau\mathbf{E} \tag{16}$$

$$\mathbf{E}'_\alpha:=\mathbf{J}'_\alpha\mathbf{E},\quad \mathbf{E}'_\beta:=\mathbf{J}'_\beta\mathbf{E},\quad \mathbf{E}'_\tau:=\mathbf{J}'_\tau\mathbf{E} \tag{17}$$

These data matrices have the structure

$$\mathbf{E}_\alpha=\mathbf{F}_\alpha\mathbf{T},\quad \mathbf{E}_\beta=\mathbf{F}_\beta\mathbf{T},\quad \mathbf{E}_\tau=\mathbf{F}_\tau\mathbf{T} \tag{18}$$

$$\mathbf{E}'_\alpha=\mathbf{F}_\alpha\Psi\mathbf{T},\quad \mathbf{E}'_\beta=\mathbf{F}_\beta\Phi\mathbf{T},\quad \mathbf{E}'_\tau=\mathbf{F}_\tau\Upsilon\mathbf{T} \tag{19}$$

Finally, we can obtain $\mathbf{E}^\dagger_\alpha\mathbf{E}'_\alpha=\mathbf{T}^{-1}\Psi\mathbf{T},\mathbf{E}^\dagger_\beta\mathbf{E}'_\beta=\mathbf{T}^{-1}\Phi\mathbf{T}$ and $\mathbf{E}^\dagger_\tau\mathbf{E}'_\tau=\mathbf{T}^{-1}\Upsilon\mathbf{T}$. This is a joint diagonalization problem [2,3].

B. MTST-MUSIC Algorithm

The 3D JADE-ESPRIT takes advantage of the Vandermonde structure of the data covariance matrices, if two or more rays have two close parameters simultaneously, this algorithm may not work properly. TST-MUSIC algorithm only deals with the problem with one close parameter. In this section we present a novel MTST-MUSIC to estimate 2-D DOA and delay. To simplify the algorithm description, we first assume that there are two or more rays with close angle α and delay simultaneously and distinct angle β. We define the covariance matrix of $\mathbf{X}_t^{(n)}$ as $\mathbf{R}^\tau$ from (9), expressed as

$$\mathbf{R}^\tau=\varepsilon\{\mathbf{X}_t^{(n)T}\mathbf{X}_t^{(n)*}\}=\mathbf{G}(\tau)P\mathbf{G}^H(\tau)+\sigma_n^2\cdot\mathbf{I} \tag{20}$$

By applying T-MUSIC to (20), it is obtained that the temporal filtering matrices $\mathbf{U}_i^t=\mathbf{I}-\mathbf{g}(\hat{t}_i)\cdot\mathbf{g}(\hat{t}_i)^H$. To simplify the algorithm description, it is assumed that there are only two estimates $\hat{t}_i(i=1,2)$ of delay. As a result, matrix $\mathbf{X}_i^t=\mathbf{X}_t^{(n)}\cdot\mathbf{U}_i^t$ can thus be generated, matrix $\mathbf{X}_i^t$ comprises information of signals which have close delay. Obviously we can obtain $\text{vec}(\mathbf{X}_i^t)=(\mathbf{U}_i^{tT}\mathbf{G}(\tau))\circ\mathbf{A}(\beta)\circ\mathbf{A}(\alpha)\beta(n)+\text{vec}(\mathbf{N}\cdot\mathbf{U}_i^t)$ and define $f_1(\mathbf{X}_i^t):=\text{unvec}_{D\times(D\times NP)}\{\text{vec}(\mathbf{X}_i^t)\}$, thus we have

$$f_1(\mathbf{X}_i^t)=\mathbf{A}(\alpha)\mathbf{B}^{(n)}\{(\mathbf{U}_i^{tT}\mathbf{G}(\tau))\circ\mathbf{A}(\beta)\}^T+f_1(\mathbf{N}\cdot\mathbf{U}_i^t) \tag{21}$$

where $f_1(\mathbf{N}\cdot\mathbf{U}_i^t)$ represent the noise term constructed from $\mathbf{N}\cdot\mathbf{U}_i^t$ in a similar way as $f_1(\mathbf{X}_i^t)$ is obtained from $\mathbf{X}_i^t$. Similarly we also define the covariance matrix of $\mathbf{X}_t^{(n)}$ as $\mathbf{R}^\alpha$, expressed as

$$\mathbf{R}^\alpha=\varepsilon\{f_1(\mathbf{X}_i^t)f_1^H(\mathbf{X}_i^t)\}=\mathbf{A}(\alpha)P\mathbf{A}^H(\alpha)+\sigma_n^2\cdot\mathbf{I} \tag{22}$$

By applying S-MUSIC to (22), it is obtained that the spatial beamforming matrices $\mathbf{U}_j^\alpha=\mathbf{I}-\mathbf{a}(\hat{\alpha}_j)\cdot\mathbf{a}(\hat{\alpha}_j)^H$, it is assumed that there are only two estimates $\hat{\alpha}_j(j=1,2)$ of angle α. As a result, matrix $\mathbf{X}_j^\alpha=\mathbf{U}_j^\alpha\cdot f_1(\mathbf{X}_i^t)$ can thus be obtained. Similarly we define matrix $f_2(\mathbf{X}_j^\alpha)=\text{unvec}_{(D\times D)\times PN}\{\text{vec}(\mathbf{X}_j^\alpha)\}$, thus have

$$f_2(\mathbf{X}_j^\alpha)=\mathbf{A}(\beta)\circ(\mathbf{U}_j^\alpha\cdot\mathbf{A}(\alpha))\mathbf{B}^{(n)}\mathbf{G}^T(\tau)\mathbf{U}_i^t+f_2(\mathbf{U}_j^\alpha\cdot f_1(\mathbf{N}\cdot\mathbf{U}_i^t)) \tag{23}$$

Where $f_2(\mathbf{X}_j^\alpha)$ comprises information of several signals which have close angle α. In (2), if we redefine the $\tilde{\mathbf{X}}(t)$ as

$$\tilde{\mathbf{X}}(t)=[\tilde{\mathbf{x}}_{1m}(t),\tilde{\mathbf{x}}_{2m}(t),\cdots,\tilde{\mathbf{x}}_{mD}(t)] \tag{24}$$

where $\tilde{\mathbf{x}}_{jm}(t)=[x_{j1}(t),\cdots,x_{jD}(t)]^T$. Substituting (24) into (8) yields

$$\text{vec}(\tilde{\mathbf{X}}(t))=(\mathbf{A}(\alpha)\circ\mathbf{A}(\beta))\mathbf{S}(t)+\text{vec}(\tilde{\mathbf{N}}(t)) \tag{25}$$

and from (23) we can obtain

$$f_3(\mathbf{X}_j^\alpha)=(\mathbf{U}_j^\alpha\cdot\mathbf{A}(\alpha))\circ\mathbf{A}(\beta)\mathbf{B}^{(n)}\mathbf{G}^T(\tau)\mathbf{U}_i^t+f_3(\mathbf{U}_j^\alpha\cdot f_1(\mathbf{N}\cdot\mathbf{U}_i^t)) \tag{26}$$

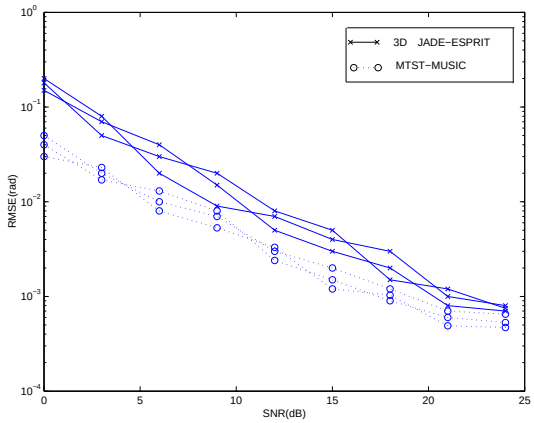

Fig. 2. Comparison of the RMSE of angle α estimates based on the 3D JADE-ESPRT, and the MTST-MUSIC algorithms.

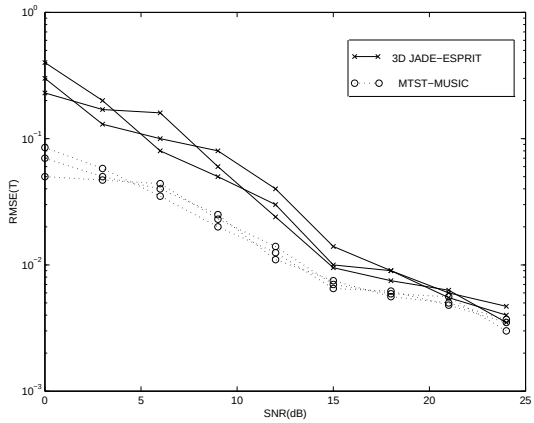

Fig. 3. Comparison of the RMSE of delay estimates based on the 3D JADE-ESPRT, and the MTST-MUSIC algorithms.

The a column of $f_3(\mathbf{X}_j^\alpha)$ are construct from that of $f_2(\mathbf{X}_j^\alpha)$ in a similar way as $\text{vec}(\tilde{\mathbf{X}}(t))$ is obtained from $\text{vec}(\mathbf{X}(t))$. Obviously from (26) we can obtain

$$f_4(\mathbf{X}_j^\alpha) = \mathbf{A}(\beta)\mathbf{B}^{(n)}\{(\mathbf{U}_i^{tT}\mathbf{G}(\tau)) \circ (\mathbf{U}_j^\alpha \mathbf{A}(\alpha))\}^T + f_4(\mathbf{U}_j^\alpha \cdot f_1(\mathbf{N} \cdot \mathbf{U}_i^t)) \tag{27}$$

where $f_4(\mathbf{X}_j^\alpha)$ is constructed from $f_3(\mathbf{X}_j^\alpha)$ in a similar way as $f_1(\mathbf{X}_i^t)$ is obtained from $\mathbf{X}_i^t$. $\mathbf{R}^\beta$ expressed as

$$\mathbf{R}^\beta := \varepsilon\{f_4(\mathbf{X}_j^\alpha) \cdot f_4^H(\mathbf{X}_j^\alpha)\} = \mathbf{A}(\beta)P\mathbf{A}^H(\beta) + \sigma^2 \cdot \mathbf{I} \tag{28}$$

with the assumption that there are two or more rays with close angle α and delay simultaneously and distinct angle β, the $\mathbf{R}^\beta$ comprises information of signals with distinct angle β and the β can accurately estimated. so angle α and delay τ are isolated and estimated.

4. Simulation Results

Some simulations are conducted in this section to verify the proposed methods. The sensor displacement is taken to be half the wavelength of the signal waves. Three narrowband signals are transmitted ($Q = 3$), the uncorrelated signal sources with identical powers while the additive noises are white Gaussian processes, and received by a 36-elements RPA ($D = 6$). The received signal $\mathbf{X}_t$ is sampled during 20 data bursts. The oversampling factor $P = 2$, and the average fading amplitudes of the three rays are equal and normalized to dB with randomly selected but constant fading phases. In the basic setup, we let the α and β be $[45°, 17°, 20°]$ and $[70°, 135°, 72°]$ respectively, and the propagation delays to be [0.90, 0.02, 0.08]T, where $T = 3.68\mu s$ is the symbol period, the number of snapshots at each sensor is $M = 1000$.

In Fig.2-3 two algorithms are carried out for comparison of the root mean square error (RMSE), including the 3D JADE-ESPRIT algorithm and modified TST-MUSIC algorithm. For each specific SNR, 300 Monte Carlo trials are conduced. As we can observe from Fig.2-3 , the modified TST-MUSIC algorithm outperforms the 3D JADE-ESPRIT algorithm.

5. Conclusion

We have presented two algorithms, 3D JADE-ESPRIT and modified TST-MUSIC, for joint 2-D angle and delay estimation using rectangular planar array in this paper. 3D JADE-ESPRIT is poor in accuracy estimation, while the modified TST-MUSIC outperforms 3D JADE-ESPRIT and the pairing of 2-D angles and delays is automatically determined. Simulation results demonstrate the validity of the suggested algorithms.

References

[1] M. C. Vanderveen, A. J. van der Veen, and A. Paulraj, ”Estimation of multipath parameters in wireless communications,” IEEE Trans. Signal Processing, vol.46, pp. 682-690, Mar. 1998.

[2] A. J. van der Veen, M. C. Vanderveen, and A. Paulraj, ”Joint angle and delay estimation (JADE) using shift-invariance properties,” IEEE Trans. Signal Processing, vol.46, pp. 405-418, Feb.1998.

[3] A. J. van der Veen, M. C. Vanderveen, and A. Paulraj, ”Joint angle and delay estimation (JADE) using shift-invariance techniques,” IEEE Signal Processing Lett.,pp. 142-145, May 1997.

[4] Y.-Y. Wang, J.-T. Chen and W.-H. Fang, ”TST-MUSIC for joint DOA-delay estimation,”IEEE Trans. Signal Processing, vol.46,pp.721-729,Apr.2001.

[5] C.G.Khatri and C.R.Rao, ”Solutions to some functional equations and their applications to characterizations of probabilitydistributions,” sankhya, vol.30, pp. 167-180, 1968.