# A FORMULATION FOR ELECTROMAGNETIC SCATTERING BY A TWO-DIMENSIONAL PERIODIC ARRAY OF ANISOTROPIC CYLINDRICAL LAYERED OBJECTS

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A periodic array of cylindrical objects is widely used as the wavelength selective or polarization selective components in microwave, millimeter-wave and optical wave region. There have been extensive theoretical investigations on the electromagnetic scattering by the periodic array during the past few decades. In this paper, the scattering from a two-dimensional periodic array of anisotropic cylindrical layered objects is investigated. The formulation process is similar to that of the Fourier series expansion method [2, 3]. We consider a set of layers uniform in the direction parallel to the axes of cylinders. The electromagnetic fields are expanded according to Floquet's theorem, and the wave propagation in each layer is described using a matrix algebra. On the other hand, the boundary condition between the layers can be fulfilled by equating the expansion coefficients associated with the tangential components of electromagnetic fields since we use the common basis for all layers.

The structure under consideration is schematically shown in Fig. 1. The cylindrical objects with radius $a$ are placed periodically in a surrounding isotropic media with permittivity $\varepsilon_0$ and permeability $\mu_0$ so that the axes of the cylinders are parallel to the $z$-axis. The periodicity axes $x$ and $u$ with the angle $\alpha$ are taken papendicular to the $z$-axis, and then the primitive cell of the periodicity is parallelogram with the side lengths of $l_x$ and $l_u$. Each cylindrical objects has a layered structure consisting of $M$ anisotropic media in which the thickness of the $\nu$th-layer ($\nu = 1, \cdots, M$) is $z_\nu - z_{\nu-1}$. Each layer is uniform in the $z$-direction, and the permittivity

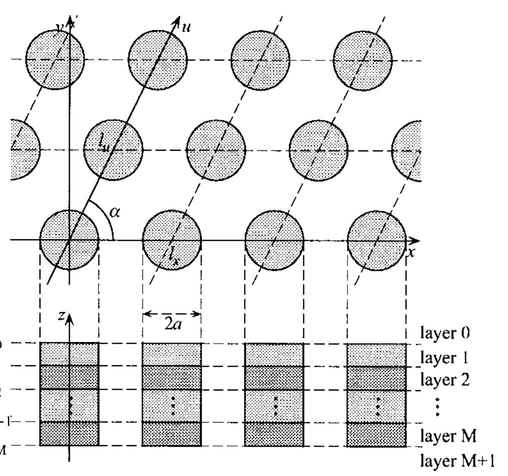


Figure 1: Top and side views of the two-dimensional periodic array of anisotropic cylindrical objects under consideration.

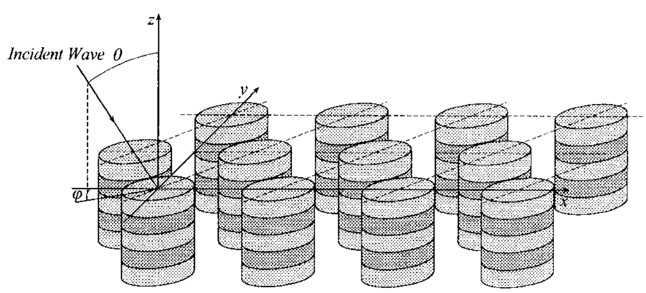


Figure 2: Incident wave.

tensor distribution $\overline{\overline{\boldsymbol{\varepsilon}}}^{(\nu)}(x,y)$ and the permeability tensor distribution $\overline{\overline{\boldsymbol{\mu}}}^{(\nu)}(x,y)$ of the layer $\nu$ $(z_{\nu-1} < z < z_\nu)$ can be expressed by the following matrices:

$$\overline{\overline{\boldsymbol{\varepsilon}}}^{(\nu)}(x,y) = \begin{pmatrix} \varepsilon_{xx}^{(\nu)}(x,y) & \varepsilon_{xy}^{(\nu)}(x,y) & \varepsilon_{xz}^{(\nu)}(x,y) \\ \varepsilon_{yx}^{(\nu)}(x,y) & \varepsilon_{yy}^{(\nu)}(x,y) & \varepsilon_{yz}^{(\nu)}(x,y) \\ \varepsilon_{zx}^{(\nu)}(x,y) & \varepsilon_{zy}^{(\nu)}(x,y) & \varepsilon_{zz}^{(\nu)}(x,y) \end{pmatrix} \tag{1}$$

$$\overline{\overline{\boldsymbol{\mu}}}^{(\nu)}(x,y) = \begin{pmatrix} \mu_{xx}^{(\nu)}(x,y) & \mu_{xy}^{(\nu)}(x,y) & \mu_{xz}^{(\nu)}(x,y) \\ \mu_{yx}^{(\nu)}(x,y) & \mu_{yy}^{(\nu)}(x,y) & \mu_{yz}^{(\nu)}(x,y) \\ \mu_{zx}^{(\nu)}(x,y) & \mu_{zy}^{(\nu)}(x,y) & \mu_{zz}^{(\nu)}(x,y) \end{pmatrix} \tag{2}$$

in the Cartesian coordinate system. We view the scattering characteristics for the incident plane wave with the incident angles $\theta$ $(0 < \theta < \pi/2)$ and $\varphi$ $(0 \le \varphi < 2\pi)$ as shown in Fig. 2 and the time dependence of $e^{j\omega t}$.

According to Floquet's theorem, the electromagnetic fields can be expressed in terms of space harmonic waves as follows:

$$e_p(x,y,z) = \boldsymbol{\psi}^t(x,y)\,\widetilde{\boldsymbol{e}}_p(z), \qquad h_p(x,y,z) = \boldsymbol{\psi}^t(x,y)\,\widetilde{\boldsymbol{h}}_p(z) \qquad (p = x,y) \tag{3}$$

with

$$\left(\boldsymbol{\psi}(x,y)\right)_m = \psi_m(x,y) = \frac{1}{\sqrt{l_x\,l_u\,\sin\alpha}}\,e^{-j(\sigma_x(m)\,x+\sigma_y(m)\,y)} \tag{4}$$

$$\sigma_x(m) = k_0\,\sin\theta\,\cos\varphi + n_x(m)\,\frac{2\pi}{l_x} \tag{5}$$

$$\sigma_y(m) = k_0\,\sin\theta\,\sin\varphi - n_x(m)\,\frac{2\pi}{l_x}\,\cot\alpha + n_u(m)\,\frac{2\pi}{l_u}\,\mathrm{cosec}\,\alpha \tag{6}$$

where $\widetilde{\boldsymbol{e}}_p(z)$ and $\widetilde{\boldsymbol{h}}_p(z)$ are expansion coefficient vectors of the $p$-components $(p = x,y)$ of electric and magnetic fields, $\boldsymbol{\psi}(x,y)$ is the orthonormal expansion basis, $(\cdot)_m$ denotes the $m$-th component of a vector, and $k_0$ is the wavenumber in free space. $n_x(m)$ and $n_u(m)$ is an integer function set which is uniquely determined by the integer $m$, and they indicate the orders of a space harmonic wave in the $x$ and $u$ direction, respectively. For numerical calculations, the infinite series expansion (3) have to be approximated by truncated expansion. We consider the space harmonic waves of $-N_x,\cdots,N_x$th-orders for the $x$-direction and $-N_u,\cdots,N_u$th-orders for the $u$-direction, and then we use the following integer functions:

$$n_x(m) = \mathrm{quotient}(m-1, 2\,N_u+1) - N_x, \qquad n_u(m) = \mathrm{mod}(m-1, 2\,N_u+1) - N_u \tag{7}$$

where quotient$(m,n)$ gives the integer quotient of $m$ and $n$ and mod$(m,n)$ gives the remainder on division of $m$ by $n$. Equation (3) is substituted into Maxwell's equations for the layer $\nu$ $(\nu = 1,\cdots,M)$. Utilizing the orthonormality among the basis functions, we obtain a set of linear equations for the expansion coefficients as follows:

$$\frac{d}{dz}\widetilde{\boldsymbol{f}}(z) = -j\,\boldsymbol{C}^{(\nu)}\,\widetilde{\boldsymbol{f}}(z) \tag{8}$$

with

$$\widetilde{\boldsymbol{f}} = \begin{pmatrix} \widetilde{\boldsymbol{e}}_x & \widetilde{\boldsymbol{e}}_y & \widetilde{\boldsymbol{h}}_x & \widetilde{\boldsymbol{h}}_y \end{pmatrix}^t, \qquad \boldsymbol{C}^{(\nu)} = \begin{pmatrix} \boldsymbol{C}_{11}^{(\nu)} & \boldsymbol{C}_{12}^{(\nu)} & \boldsymbol{C}_{13}^{(\nu)} & \boldsymbol{C}_{14}^{(\nu)} \\ \boldsymbol{C}_{21}^{(\nu)} & \boldsymbol{C}_{22}^{(\nu)} & \boldsymbol{C}_{23}^{(\nu)} & \boldsymbol{C}_{24}^{(\nu)} \\ \boldsymbol{C}_{31}^{(\nu)} & \boldsymbol{C}_{32}^{(\nu)} & \boldsymbol{C}_{33}^{(\nu)} & \boldsymbol{C}_{34}^{(\nu)} \\ \boldsymbol{C}_{41}^{(\nu)} & \boldsymbol{C}_{42}^{(\nu)} & \boldsymbol{C}_{43}^{(\nu)} & \boldsymbol{C}_{44}^{(\nu)} \end{pmatrix} \tag{9}$$

with

$$C_{11}^{(\nu)} = -D_x A_{zz}^{(\nu)^{-1}} A_{zx}^{(\nu)} - B_{yz}^{(\nu)} B_{zz}^{(\nu)^{-1}} D_y \qquad C_{31}^{(\nu)} = -\omega A_{yx}^{(\nu)} + \omega A_{yz}^{(\nu)} A_{zz}^{(\nu)^{-1}} A_{zx}^{(\nu)} - \frac{1}{\omega} D_x B_{zz}^{(\nu)^{-1}} D_y$$

$$C_{12}^{(\nu)} = -D_x A_{zz}^{(\nu)^{-1}} A_{zy}^{(\nu)} + B_{yz}^{(\nu)} B_{zz}^{(\nu)^{-1}} D_x \qquad C_{32}^{(\nu)} = -\omega A_{yy}^{(\nu)} + \omega A_{yz}^{(\nu)} A_{zz}^{(\nu)^{-1}} A_{zy}^{(\nu)} + \frac{1}{\omega} D_x B_{zz}^{(\nu)^{-1}} D_x$$

$$C_{13}^{(\nu)} = \frac{1}{\omega} D_x A_{zz}^{(\nu)^{-1}} D_y + \omega B_{yx}^{(\nu)} - \omega B_{yz}^{(\nu)} B_{zz}^{(\nu)^{-1}} B_{zx}^{(\nu)} \qquad C_{33}^{(\nu)} = -A_{yz}^{(\nu)} A_{zz}^{(\nu)^{-1}} D_y - D_x B_{zz}^{(\nu)^{-1}} B_{zx}^{(\nu)}$$

$$C_{14}^{(\nu)} = -\frac{1}{\omega} D_x A_{zz}^{(\nu)^{-1}} D_x + \omega B_{yy}^{(\nu)} - \omega B_{yz}^{(\nu)} B_{zz}^{(\nu)^{-1}} B_{zy}^{(\nu)} \qquad C_{34}^{(\nu)} = A_{yz}^{(\nu)} A_{zz}^{(\nu)^{-1}} D_x - D_x B_{zz}^{(\nu)^{-1}} B_{zy}^{(\nu)}$$

$$C_{21}^{(\nu)} = -D_y A_{zz}^{(\nu)^{-1}} A_{zx}^{(\nu)} + B_{xz}^{(\nu)} B_{zz}^{(\nu)^{-1}} D_y \qquad C_{41}^{(\nu)} = \omega A_{xx}^{(\nu)} - \omega A_{xz}^{(\nu)} A_{zz}^{(\nu)^{-1}} A_{zx}^{(\nu)} - \frac{1}{\omega} D_y B_{zz}^{(\nu)^{-1}} D_y$$

$$C_{22}^{(\nu)} = -D_y A_{zz}^{(\nu)^{-1}} A_{zy}^{(\nu)} - B_{xz}^{(\nu)} B_{zz}^{(\nu)^{-1}} D_x \qquad C_{42}^{(\nu)} = \omega A_{xy}^{(\nu)} - \omega A_{xz}^{(\nu)} A_{zz}^{(\nu)^{-1}} A_{zy}^{(\nu)} + \frac{1}{\omega} D_y B_{zz}^{(\nu)^{-1}} D_x$$

$$C_{23}^{(\nu)} = \frac{1}{\omega} D_y A_{zz}^{(\nu)^{-1}} D_y - \omega B_{xx}^{(\nu)} + \omega B_{xz}^{(\nu)} B_{zz}^{(\nu)^{-1}} B_{zx}^{(\nu)} \qquad C_{43}^{(\nu)} = A_{xz}^{(\nu)} A_{zz}^{(\nu)^{-1}} D_y - D_y B_{zz}^{(\nu)^{-1}} B_{zx}^{(\nu)}$$

$$C_{24}^{(\nu)} = -\frac{1}{\omega} D_y A_{zz}^{(\nu)^{-1}} D_x - \omega B_{xy}^{(\nu)} + \omega B_{xz}^{(\nu)} B_{zz}^{(\nu)^{-1}} B_{zy}^{(\nu)} \qquad C_{44}^{(\nu)} = -A_{xz}^{(\nu)} A_{zz}^{(\nu)^{-1}} D_x - D_y B_{zz}^{(\nu)^{-1}} B_{zy}^{(\nu)}$$

$$\left(A_{pq}^{(\nu)}\right)_{mn} = \int_0^{l_y}\int_0^{l_x} \varepsilon_{pq}^{(\nu)}(x,y)\,\psi_m^*(x,y)\,\psi_n(x,y)\,dx\,dy \qquad \left(B_{pq}^{(\nu)}\right)_{mn} = \int_0^{l_y}\int_0^{l_x} \mu_{pq}^{(\nu)}(x,y)\,\psi_m^*(x,y)\,\psi_n(x,y)\,dx\,dy$$

$$\left(D_p\right)_{mn} = \sigma_p(m)\,\delta_{mn}$$

where $p,q = x,y,z$, $(\cdot)_{mn}$ denotes the $(m,n)$-component of a matrix, $\delta_{mn}$ is Kronecker's delta, and the asterisk denotes the complex conjugate.

The eigenvalues $\beta_m^{(\nu)}$ of matrix $C^{(\nu)}$ and the associated eigenvectors $p_m^{(\nu)}$ determine the propagation constants for the $z$-direction and the field distribution of the eigenmodes, respectively. The solutions to Eq. (8) are expressed as

$$\widetilde{f}(z) = P^{(\nu)}\,U^{(\nu)}(z - z_{\nu-1})\,P^{(\nu)^{-1}}\,\widetilde{f}(z_{\nu-1}) \tag{10}$$

with

$$\left(U^{(\nu)}(z)\right)_{mn} = e^{-j\,\beta_m^{(\nu)}\,z}\,\delta_{mn}, \qquad P^{(\nu)} = \begin{pmatrix} p_1^{(\nu)} & p_2^{(\nu)} & \cdots \end{pmatrix}. \tag{11}$$

This equation gives the representation of the electromagnetic wave propagation in each uniform layer, and yields the relation between $\widetilde{f}(z_\nu)$ and $\widetilde{f}(z_{\nu-1})$ which determine the tangential components of electromagnetic fields at the upper and lower surface of the layer $\nu$. On the other hand, we should note that the expansion coefficient vector $\widetilde{f}(z)$ is continuous at the boundaries between the layers, because we use common basis for all layers and $\widetilde{f}(z)$ consists of the expansion coefficients of the tangential components of electromagnetic fields. The relation (10) is successively applied for all layers, and then we can derive an equation that relates the expansion coefficients at the upper end ($z = z_0$) to those at the lower end ($z = z_M$) as follows:

$$\widetilde{f}(z_M) = F\,\widetilde{f}(z_0) \tag{12}$$

with

$$F = F^{(M)}\,F^{(M-1)}\,\cdots\,F^{(2)}\,F^{(1)}, \qquad F^{(\nu)} = P^{(\nu)}\,U^{(\nu)}(z_\nu - z_{\nu-1})\,P^{(\nu)^{-1}}. \tag{13}$$

The layers 0 ($z > z_0$) and $M+1$ ($z < z_M$) are isotropic and homogeneous, and therefore the space harmonic waves are not coupled to each other. Then, we can easily calculate the matrices $P^{(0)}$ and $P^{(M+1)}$ and obtain the analytical expressions. We chose these matrices so as to satisfy the following relation:

$$\begin{pmatrix} a_{d,s}^{(\nu)}(z) & a_{d,p}^{(\nu)}(z) & a_{u,s}^{(\nu)}(z) & a_{u,p}^{(\nu)}(z) \end{pmatrix}^t = P^{(\nu)^{-1}}\,\widetilde{f}(z) \quad (\nu = 0, M+1) \tag{14}$$

where $\boldsymbol{a}_{d,s}^{(\nu)}$, $\boldsymbol{a}_{d,p}^{(\nu)}$, $\boldsymbol{a}_{u,s}^{(\nu)}$, and $\boldsymbol{a}_{u,p}^{(\nu)}$ indicate the amplitude vectors of the $s$-polarized downward, $p$-polarized downward, $s$-polarized upward, and $p$-polarized upward waves, respectively. Using Eq. (14), we obtain the following relation from Eq. (15):

$$\begin{pmatrix} \boldsymbol{a}_{d,s}^{(M+1)}(z_M) \\ \boldsymbol{a}_{d,p}^{(M+1)}(z_M) \\ \boldsymbol{a}_{u,s}^{(M+1)}(z_M) \\ \boldsymbol{a}_{u,p}^{(M+1)}(z_M) \end{pmatrix} = \boldsymbol{K} \begin{pmatrix} \boldsymbol{a}_{d,s}^{(0)}(z_0) \\ \boldsymbol{a}_{d,p}^{(0)}(z_0) \\ \boldsymbol{a}_{u,s}^{(0)}(z_0) \\ \boldsymbol{a}_{u,p}^{(0)}(z_0) \end{pmatrix} \tag{15}$$

with

$$\boldsymbol{K} = \boldsymbol{P}^{(M+1)^{-1}} \boldsymbol{F}\, \boldsymbol{P}^{(0)}. \tag{16}$$

In this equation, $\boldsymbol{a}_{d,s}^{(0)}(z_0)$, $\boldsymbol{a}_{d,p}^{(0)}(z_0)$, $\boldsymbol{a}_{u,s}^{(M+1)}(z_M)$, and $\boldsymbol{a}_{u,p}^{(M+1)}(z_M)$ give the amplitudes of incoming waves. Since we consider that the plane wave is incident from the layer 0 ($z > z_0$), the components of $\boldsymbol{a}_{d,s}^{(0)}(z_0)$ and $\boldsymbol{a}_{d,p}^{(0)}(z_0)$ except for those associated with the fundamental space harmonic waves are zero. On the other hand, there is no reflection from the layer $M+1$ ($z < z_M$), and then the boundary conditions are given by $\boldsymbol{a}_{u,s}^{(M+1)}(z_M) = \boldsymbol{a}_{u,p}^{(M+1)}(z_M) = \boldsymbol{0}$. These conditions are used in Eq. (15) to determine the amplitudes $\boldsymbol{a}_{u,s}^{(0)}(z_0)$ and $\boldsymbol{a}_{u,p}^{(0)}(z_0)$ for the reflected waves and $\boldsymbol{a}_{d,s}^{(M+1)}(z_M)$ and $\boldsymbol{a}_{d,p}^{(M+1)}(z_M)$ for the transmitted waves.

We have presented a numerical approach for the electromagnetic scattering from a two-dimensional periodic array of anisotropic cylindrical layered objects. The formulation is based on Floquet's theorem, and the electromagnetic fields are expanded by the space harmonic waves. The common expansion basis is used for all layers and the boundary condition between the layers are satisfied by equating the expansion coefficients. This greatly simplifies the numerical procedure and yields wide applicability.

## References


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