Some Analytic Formulations of Weakly Singular Integrals over Polygon for IPO Applications

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Abstract - **To treat weakly singular integrals, analytic formulations have been derived for the near-field correction of the iterative physical optics applications. They are analytically derived for a flat polygon patch based on the Stokes' theorem and numerically verified.**

Index Terms — **Weakly singular integrals, Iterative physical optics, Near-field correction.**

1. Introduction

The iterative physical optics (IPO) method has been applied to various electromagnetics problems. To improve the accuracy of the IPO, the treatment for singular integral kernels is essential. The singular integral kernels arise from the Green's function, which are mainly classified as $1/R^3$ hypersingular integrals (HSIs), $1/R^2$ strongly singular integrals (SSIs), and $1/R$ weakly singular integrals (WSIs). In this work, we focus on the formulations of WSIs. One of the most widely used techniques to treat WSIs is the Duffy's method [1]. However, it increases the number of numerical computations and is less accurate for near-singular cases.

In this work, some analytic formulations for WSIs are derived for a flat polygon patch based on the Stokes' theorem [2] and also verified numerically.

2. Near-field Correction for IPO

The IPO iteratively updates the surface currents on an object to calculate the multiple interaction. To update the IPO currents following surface integral should be computed.

$$\vec{E} = -\frac{jZ_0}{4\pi k_0}\iint_{\Delta S}\left[G_1\vec{J}_e + G_2\vec{R}\left(\vec{R}\cdot\vec{J}_e\right) + \frac{k_0}{jZ_0}G_3\left(\vec{J}_m\times\vec{R}\right)\right]dS\,, \quad (1)$$

where $\vec{R} = (u_0-u)\hat{u} + (v_0-v)\hat{v} + (w_0-w)\hat{w}$, $G_1 = ({k_0}^2R^2 - 1 - jk_0R)e^{-jk_0R}/R^3$, $G_2 = ({k_0}^2R^2 + 3 + 3jk_0R)e^{-jk_0R}/R^5$, $G_3 = (1 + jk_0R)e^{-jk_0R}/R^3$, ΔS is the triangular source patch. Accurately to calculate (1) for the near-singular case, the singularity subtraction method is usually used. For example, the second term in (1) can be treated as

$$\begin{aligned}&\iint_{\Delta S} G_2\vec{R}\left(\vec{R}\cdot\vec{J}_e\right)dS\\ &\approx \sum_{i=1}^{N_p}\left[\iint_{\Delta S_i}\left(G_2 - \frac{3}{R^5} - \frac{{k_0}^2}{2R^3} - \frac{{k_0}^4}{8R}\right)\vec{R}\vec{R}d\Omega\right]\cdot\vec{J}_{e,i}\\ &+\sum_{i=1}^{N_p}\left[\iint_{\Delta S_i}\left(\frac{3}{R^5} + \frac{{k_0}^2}{2R^3} + \frac{{k_0}^4}{8R}\right)\vec{R}\vec{R}d\Omega\right]\cdot\vec{J}_{e,i}\end{aligned} \quad (2)$$

where N_p is the number of the subdivided polygons. The surface current can be approximated as constant $\vec{J}_{e,i}$ on i th polygon, and the singular terms are subtracted by the Taylor series expansion. The first integral terms can be numerically calculated as it is regular. The second singular integral terms should be calculated by other means. Among the singular integral terms, only WSIs are considered in this work.

3. Derivation of Formulas for WSIs

Based on the Stokes' theorem for surface integrations over a polygon patch, we can formulate following equations [2]

$$\iint_{\Delta S}\frac{\partial A_u}{\partial v} = -\oint_C A_u du;\ \iint_{\Delta S}\frac{\partial A_v}{\partial u} = \oint_C A_v dv\,, \quad (3)$$

where $\vec{A} = A_u\hat{u} + A_v\hat{v} + A_w\hat{w}$ is a vector defined over a surface patch ΔS in a local coordinate (u,v,w) , and C is the boundary of ΔS . The patch plane are assumed to be on uv plane and $\hat{w} = \hat{n}$, i.e. $n_u = 0$, $n_v = 0$, and $n_w = 1$.

The subtracted WSIs in (2) are as follows

$$\begin{aligned}&I_1 = \iint_{\Delta S}\frac{1}{R}dS;\ I_2 = \iint_{\Delta S}\frac{(u_0-u)^2}{R^3}dS;\ I_3 = \iint_{\Delta S}\frac{(v_0-v)^2}{R^3}dS;\\ &I_4 = \iint_{\Delta S}\frac{(u_0-u)(v_0-v)}{R^3}dS;\ I_5 = \iint_{\Delta S}\frac{(u_0-u)^2}{R}dS;\\ &I_6 = \iint_{\Delta S}\frac{(v_0-v)^2}{R}dS;\ I_7 = \iint_{\Delta S}\frac{(u_0-u)}{R}dS;\\ &I_8 = \iint_{\Delta S}\frac{(v_0-v)}{R}dS;\ I_9 = \iint_{\Delta S}\frac{(u_0-u)(v_0-v)}{R}dS\end{aligned} \quad (4)$$

where the observation points (u_0,v_0,w_0) is closely located not inside but outside of the N -sided source polygon patch whose vertices are $p_i(u_i,v_i,w_i)$ with $i = 1,2,\cdots,N$.

The integrals in (4) can be analytically derived based on (3) and integral identities in [3]. For instance, I_1 , I_2 , and I_4 can be respectively calculated into a closed-form expressions as

$$\begin{aligned}I_1 &= \iint_{\Delta S}\frac{1}{R}dS = \oint_C \ln\left|R-(u_0-u)\right|dv\\ &= \sum_{i=1}^{N}\begin{bmatrix}\dfrac{A_iv_0-B_i}{\sqrt{1+A_i^2}}\ln\dfrac{\left|-A_iB_i + v_{i+1} + A_i^2v_{i+1} - v_0 + R_{i+1}\sqrt{1+A_i^2}\right|}{\left|-A_iB_i + v_i + A_i^2v_i - v_0 + R_i\sqrt{1+A_i^2}\right|}\\ -w_0\tan^{-1}\left(\dfrac{a_ic_i}{a_i^2 + l_i^2\left({w_0}^2 + w_0R_{i+1}\right)}\right)\\ -w_0\tan^{-1}\left(\dfrac{a_ib_i}{a_i^2 + l_i^2\left({w_0}^2 + w_0R_i\right)}\right)\end{bmatrix}\end{aligned}$$

$$\begin{aligned}I_2 &= \iint_{\Delta S}\frac{(u_0-u)^2}{R^3}dS = \oint_C\left[\frac{(u_0-u)}{R} + \ln\left|R-(u_0-u)\right|\right]dv\\ &= \sum_{i=1}^{N}\left[-\frac{A_i}{\sqrt{1+A_i^2}}\left(R_{i+1}-R_i\right) + \left(\frac{B_i - A_iv_0}{\left(1+A_i^2\right)^{3/2}} + \frac{A_iv_0 - B_i}{\sqrt{1+A_i^2}}\right)\right]\end{aligned}$$

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$$
\left[
\begin{array}{l}
-\dfrac{A_i}{\sqrt{1+A_i^2}}\left(R_{i+1}-R_i\right)+\left(\dfrac{B_i-A_i v_0}{\left(1+A_i^2\right)^{3/2}}+\dfrac{A_i v_0-B_i}{\sqrt{1+A_i^2}}\right) \\
\times\ln\dfrac{\left|-A_iB_i+v_{i+1}+A_i^2v_{i+1}-v_0+R_{i+1}\sqrt{1+A_i^2}\right|}{\left|-A_iB_i+v_i+A_i^2v_i-v_0+R_i\sqrt{1+A_i^2}\right|} \\
-w_0\tan^{-1}\left(\dfrac{a_ic_i}{a_i^2+l_i^2\left(w_0^2+w_0R_{i+1}\right)}\right) \\
-w_0\tan^{-1}\left(\dfrac{a_ib_i}{a_i^2+l_i^2\left(w_0^2+w_0R_i\right)}\right)
\end{array}
\right]
$$

$$
\begin{aligned}
I_4 &= \iint_{\Delta S}\frac{(u_0-u)(v_0-v)}{R^3}dS=\oint_C\frac{(v_0-v)}{R}dv \\
&= \sum_{i=1}^{N}\left[
\begin{array}{l}
-\dfrac{1}{1+A_i^2}\left(R_{i+1}-R_i\right)+\dfrac{A_i\left(A_iv_0-B_i\right)}{\left(1+A_i^2\right)^{3/2}} \\
\times\ln\dfrac{\left|-A_iB_i+v_{i+1}+A_i^2v_{i+1}-v_0+R_{i+1}\sqrt{1+A_i^2}\right|}{\left|-A_iB_i+v_i+A_i^2v_i-v_0+R_i\sqrt{1+A_i^2}\right|}
\end{array}
\right]
\end{aligned}
\qquad (5)
$$

where

$$
\begin{aligned}
&A_i=\frac{u_{i+1}-u_i}{v_{i+1}-v_i},\ B_i=u_0-u_i+A_iv_i \\
&R_i=\sqrt{\left(u_0-u_i\right)^2+\left(v_0-v_i\right)^2+w_0^2} \\
&l_i=\sqrt{\left(u_{i+1}-u_i\right)^2+\left(v_{i+1}-v_i\right)^2} \\
&a_i=\left(u_i-u_0\right)\left(v_{i+1}-v_i\right)-\left(v_i-v_0\right)\left(u_{i+1}-u_i\right) \\
&b_i=\left(u_{i+1}-u_i\right)\left(u_i-u_0\right)+\left(v_{i+1}-v_i\right)\left(v_i-v_0\right) \\
&c_i=\left(u_{i+1}-u_i\right)\left(u_{i+1}-u_0\right)+\left(v_{i+1}-v_i\right)\left(v_{i+1}-v_0\right)
\end{aligned}
\qquad (6)
$$

with the assumption that $i+1=\mathrm{mod}(i,N)+1$.

4. Numerical Verification

To calculate the surface integrals numerically, the Gauss quadrature rule is widely used [4]. However, the accuracy is deteriorated when the distance between the observation point and the source patch is very close. Thus, in this work, we keep dividing a source patch into the sub-triangle patches and apply the low-order quadrature rule to each sub-triangle patches until the numerical value converges with a given certain tolerance.

A source triangle patch is located at uv plane where the vertices are defined as $p_1(0.7\lambda,0.1\lambda,0)$, $p_2(0.4\lambda,0.3\lambda,0)$, and $p_3(0.1\lambda,0.2\lambda,0)$, respectively. And the observation point is located at $p_0(0.4\lambda,0.2\lambda,w_0)$ where w_0 varies from 0.01λ to 1λ. As the number of sub-triangle patches, N_{tri} increases, the accuracy of the numerical value improves since the sampling points also increases over the triangle patch. $N_{\mathrm{Q.P.}}$ is the total number of quadrature points as shown in Fig. 1. The analytic solution of I_1 is in accurate agreement with the numerical solution with the Gauss quadrature of $N_{\mathrm{tri}}=1024$ and $N_{\mathrm{Q.P.}}=7168$ even when w_0 is very small. Based on the numerical solutions of $N_{\mathrm{tri}}=1024$ and $N_{\mathrm{Q.P.}}=7168$, the relative errors ε for the analytic formulas of $I_1 \sim I_9$ in (5) are calculated (see Fig. 2). The analytic formulas have low relative errors less than 10^{-4} and are accurate for any w_0 values.

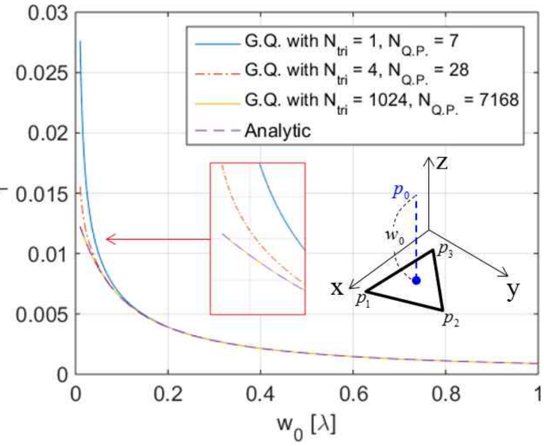

Fig. 1. Verification of I_1 with respect to w_0 by applying the Gauss quadrature rule for subdivided triangle patches.

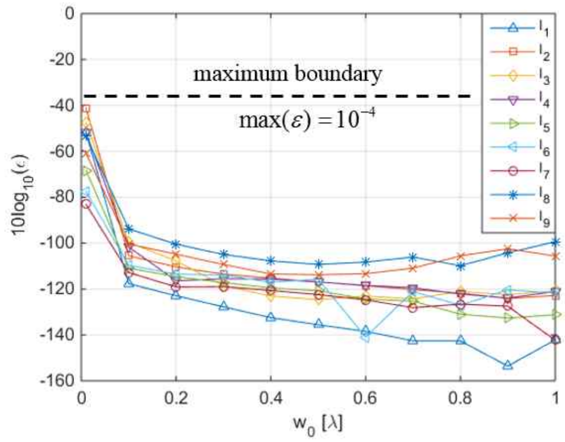

Fig. 2. Relative error ε comparison of $I_1 \sim I_9$ with respect to numerical solutions of $N_{\mathrm{tri}}=1024$ and $N_{\mathrm{Q.P.}}=7168$.

5. Conclusion

Some analytic formulas for WSIs are derived based on Stokes' theorem for a flat polygon patch and they are also verified numerically. It is not only exact for any near-singular case, but also require low computations.

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