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Numerical Dispersion of CIP Method for Electromagnetic Problems

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1 Introduction

The constrained interpolation profile (CIP) method is a computational scheme for problems including different phases[1], and it provides possibilities to reduce computational resources for solving electromagnetic problems. The method is based on the upwind scheme with the profiles between two grid points, interpolated in terms of cubic polynomials which allow us to calculate fields at the next time step with good precision. However, propagating waves suffer from numerical dispersion, like other schemes.

It is important to obatin the fomula of the numerical dispersion in order to estimate the precision of the computational results. In this paper, the numerical dispersion for the grid-aligned propagation, i.e. the propagation along the principal grid axes, is derived theoretically, and is then examined numerically. The comparison with the one of the finite-difference time-domain (FDTD) method is also performed.

2 Numerical dispersion relation of the CIP method

2.1 CIP method of the 3-rd order

We consider a wave propagating to $+x$-direction with the velocity of c_0. The field value is indicated by $f(x,t)$ and the derivative is expressed as $g(x,t) = \partial f/\partial x$. Their discretized forms are given by $f_i^n \equiv f(i\,\Delta x, n\,\Delta t)$ and $g_i^n \equiv g(i\,\Delta x, n\,\Delta t)$, where Δx and Δt are the spatial and temporal discretization, respectively. Therefore, the explicit form of the CIP updating scheme is given by the following equations:

$$f_i^{n+1} = A_1 f_i^n + A_2 f_{i-1}^n + A_3 g_i^n + A_4 g_{i-1}^n, \tag{1}$$
$$g_i^{n+1} = B_1 f_i^n + B_2 f_{i-1}^n + B_3 g_i^n + B_4 g_{i-1}^n, \tag{2}$$

where the coefficients A_α and B_α are given by

$$A_1 = 1 + 2\xi^3 - 3\xi^2, \quad A_2 = 3\xi^2 - 2\xi^3, \quad A_3 = \Delta x(2\xi^2 - \xi^3 - \xi), \quad A_4 = \Delta x(\xi^2 - \xi^3),$$
$$B_1 = \frac{6}{\Delta x}(\xi - \xi^2), \quad B_2 = -B_1, \quad B_3 = 1 + 3\xi^2 - 4\xi, \quad B_4 = 3\xi^2 - 2\xi,$$

and ξ is the so-called the Courant number, and is given by $\xi = \frac{c_0\,\Delta t}{\Delta x}$.

Consider a plane wave at an angular frequency ω in order to derive the numerical dispersion of the CIP method:

$$f_i^n = f_0 \exp(j\omega t - j\tilde{k}x), \quad g_i^n = g_0 \exp(j\omega t - j\tilde{k}x), \tag{3}$$

where f_0 and g_0 are constants, and $\tilde{k}$ is the numerical wave number. Substituting Eq. (3) into Eqs. (1) and (2), we have

$$e^{j\omega\Delta t}\begin{bmatrix} f_0 \\ g_0 \end{bmatrix} = \begin{bmatrix} A_1{+}A_2 e^{j\tilde{k}\Delta x} & A_3{+}A_4 e^{j\tilde{k}\Delta x} \\ B_1{+}B_2 e^{j\tilde{k}\Delta x} & B_3{+}B_4 e^{j\tilde{k}\Delta x} \end{bmatrix} \bullet \begin{bmatrix} f_0 \\ g_0 \end{bmatrix} = \mathbf{F}_2 \bullet \begin{bmatrix} f_0 \\ g_0 \end{bmatrix} \tag{4}$$

The dispersion relation is the condition to satisfy the above equation, that is,

$$\left| e^{j\omega\Delta t}\mathbf{I}_2 - \mathbf{F}_2 \right| = 0, \tag{5}$$

where $\mathbf{I}_m$ is the identity matrix of $m \times m$. The equation is written explicitly in the following form.

$$\{\exp(j\omega\,\Delta t) - A_1 - A_2\exp(j\tilde{k}\Delta x)\}\{\exp(j\omega\,\Delta t) - B_1 - B_2\exp(j\tilde{k}\Delta x)\}$$
$$= \{A_3 + A_4\exp(j\tilde{k}\Delta x)\}\{B_3 + B_4\exp(j\tilde{k}\Delta x)\}. \tag{6}$$

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For the analysis of the numerical dispersion, the more important parameters are the sampling density and the Courant number ξ. The sampling density D is the number of the discretization per wavelength: $D = \frac{\lambda_0}{\Delta x} = \frac{2\pi}{k_0 \Delta x}$, where λ and k_0 are the physical wavelength and the physical wavenumber. Introducing the normalized numerical wavenumber $\tilde{k}_n = \tilde{k}/k_0$, we can rewrite the numerical dispersion relation (6) into

$$\begin{aligned}\{\exp(j\xi P) - A_1 - A_2 \exp(j\tilde{k}_n P)\}\{\exp(j\xi P) - B_1 - B_2 \exp(j\tilde{k}_n P)\} \\ = \{A_3 + A_4 \exp(j\tilde{k}_n P)\}\{B_3 + B_4 \exp(j\tilde{k}_n P)\},\end{aligned} \tag{7}$$

where $P = 2\pi/D$.

The numerical dispersion relation is obtained by finding $\tilde{k}_n = \tilde{k}_n^r + j\tilde{k}_n^i$ which satisfies Eq. (7) for given ξ and D.

2.2 CIP method of the 5-th order

The straightforward extention of the CIP method is easily obtained by using the second derivatives to interpolate the profiles with the polynomials of the 5-th order. Here, we consider again the fields propagating to $+x$-direction. Thus the profile between $i\Delta x$ and $(i-1)\Delta x$ is given by

$$f(x) = \sum_{n=0}^{5} a_n (x - x_i)^n, \tag{8}$$

where the coefficients a_n's are $a_0 = f_i$, $a_1 = g_i$, $a_2 = \frac{1}{2}h_i$,

$$a_5 = \frac{6}{\Delta x^5}(f_i - f_{i-1}) - \frac{3}{\Delta x^4}(g_i + g_{i-1}) + \frac{1}{2\Delta x^3}(h_i - h_{i-1}), \tag{9}$$

$$a_4 = \frac{15}{\Delta x^4}(f_i - f_{i-1}) - \frac{1}{\Delta x^3}(8g_i + 7g_{i-1}) + \frac{1}{2\Delta x^2}(3h_i - 2h_{i-1}), \tag{10}$$

$$a_3 = \frac{10}{\Delta x^3}(f_i - f_{i-1}) - \frac{1}{\Delta x^2}(6g_i + 4g_{i-1}) + \frac{1}{2\Delta x}(3h_i - h_{i-1}), \tag{11}$$

and h_i is the second derivative at $i\Delta x$.

Thus, the updating equation for f_i, g_i, and h_i are given in the following form.

$$f_i^{n+1} = A'_1 f_i^n + A'_2 f_{i-1}^n + A'_3 g_i^n + A'_4 g_{i-1}^n + A'_5 h_i^n + A'_6 h_{i-1}^n, \tag{12}$$

$$g_i^{n+1} = B'_1 f_i^n + B'_2 f_{i-1}^n + B'_3 g_i^n + B'_4 g_{i-1}^n + B'_5 h_i^n + B'_6 h_{i-1}^n, \tag{13}$$

$$h_i^{n+1} = C'_1 f_i^n + C'_2 f_{i-1}^n + C'_3 g_i^n + C'_4 g_{i-1}^n + C'_5 h_i^n + C'_6 h_{i-1}^n, \tag{14}$$

where the coefficients A'_m, B'_m, C'_m's are written in Appendix A.

The similar analysis gives the dispersion relation: $\left| e^{j\xi P}\mathbf{I}_3 - \mathbf{F}_3 \right| = 0$, where

$$\mathbf{F}_3 = \begin{bmatrix} A'_1 + A'_2 e^{j\tilde{k}_n P} & A'_3 + A'_4 e^{j\tilde{k}_n P} & A'_5 + A'_6 e^{j\tilde{k}_n P} \\ B'_1 + B'_2 e^{j\tilde{k}_n P} & B'_3 + B'_4 e^{j\tilde{k}_n P} & B'_5 + B'_6 e^{j\tilde{k}_n P} \\ C'_1 + C'_2 e^{j\tilde{k}_n P} & C'_3 + C'_4 e^{j\tilde{k}_n P} & C'_5 + C'_6 e^{j\tilde{k}_n P} \end{bmatrix}. \tag{15}$$

The explicit equation for the dispersion relation is given in Appendix B.

2.3 Numerical results and comparison with the FDTD method

In order to verify the numerical dispersion relation derived in the above sections, we compare the waveforms computed actually by the CIP method, and the ones calculated by using the numerical dispersion[3].

We define time series p_0^n and q_0^n $(n = 0, 1, \cdots)$ to be observed as the field value and the derivative at a reference point. For the CIP computation, this times series is used as the incident wave into the analysis region. Therefore, at $x = \Delta x$ the CIP scheme is written as follows:

$$f_1^{n+1} = A_1 f_1^n + A_2 p_0^n + A_3 g_1^n + A_4 q_0^n, \quad g_1^{n+1} = B_1 g_1^n + B_2 q_0^n + B_3 f_1^n + B_4 p_0^n. \tag{16}$$

We can predict the field values and the derivatives excited by the above time series at an arbitrary point, by using the nurmerical dispersion. The spectrum of the field observed at the point x can be calculated by multiplying the one at the reference point by the propagation factor. Therefore, the time series p_i^n to be observed at the point $i\,\Delta x$ can be calculated by

$$p_i^{\{0,1,\cdots,m\}} = \mathcal{F}^{-1}\left[\mathcal{F}\left[p_0^{\{0,1,\cdots,m\}}\right] e^{-j\tilde{k}\,i\,\Delta x}\right], \tag{17}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ is the operater taking the fast Fourier transform (FFT) and the inverse FFT.

Fig. 1 shows the waveforms observed at $x = 5\Delta x$, $300\Delta x$, and $900\Delta x$. In the numerical examination, we choose the Gaussian pulse as the initial time series at the reference point $x = 0$:

$$p_0^i = \exp\left(-\left\{\frac{i\,\Delta t - t_0}{\sigma}\right\}^2\right), \quad q_0^i = \frac{2(i\,\Delta t - t_0)}{c_0\sigma^2}\exp\left(-\left\{\frac{i\,\Delta t - t_0}{\sigma}\right\}^2\right). \tag{18}$$

The used values are $\Delta t = \xi\Delta x/c$, $\xi =$, $t_0 = ...\Delta t$, $\sigma = ...\Delta t$. The solid lines in Fig. 1 are the results computed by CIP scheme, and the circles are obtained by the numerical dispersion in Eq. (7) and the FFTs. The two results agree with each other, which states that the numerical dispersion relation is appropriately derived. Fig. 2 shows the waveforms calculated by the 5-th CIP and the numerical dispersion relation. The results of the 5-th CIP shows very excellent conservation of the initial waveform after $300\Delta x$ and $900\Delta x$ propagation.

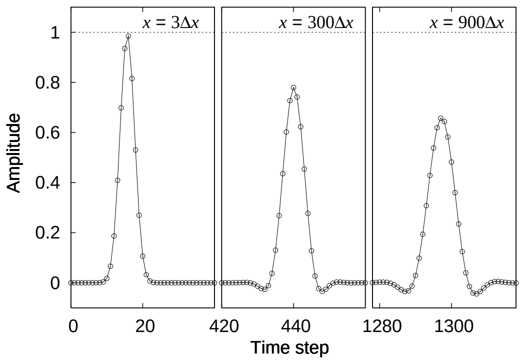

Figure 1: The waveforms calculated from the numerical dispersion and the results directly by CIP scheme.

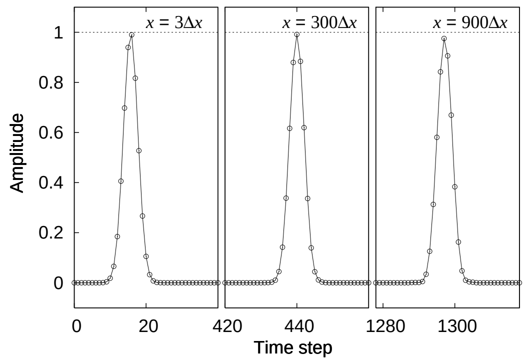

Figure 2: The waveforms for the case of the 5-th CIP.

Fig. 3 shows the numerical phase velocities of CIP and FDTD[2] as a function of the sampling density. The phase velocity is obtained by $\omega/(\tilde{k}_n^r k_0)$. The velocities are normalized by the physical velocity. The FDTD method has the relative error of 1% even for $D = 10$, while for the CIP method the numerical phase velocity is very close to the physical one down to $D = 2$. Fig. 4 shows the numerical attenuation characteristics of CIP and FDTD. The attenuation per unit cell is given by $2\pi\tilde{k}_i^n/D$. For the case of FDTD, the numerical attenuation does not exist down to ..., while CIP method has the attenuation for any D. We can also say that the 5-th CIP has smaller ficticious attenuation than the 3-rd CIP.

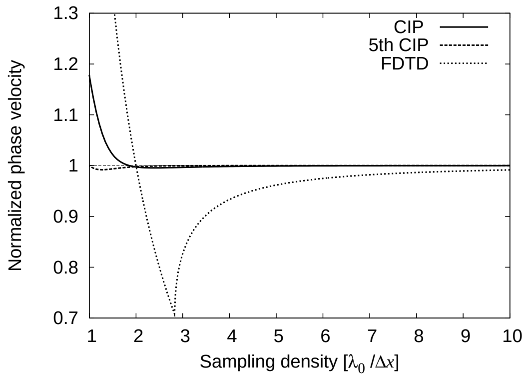

Figure 3: Normalized numerical phase velocities of CIP and FDTD.

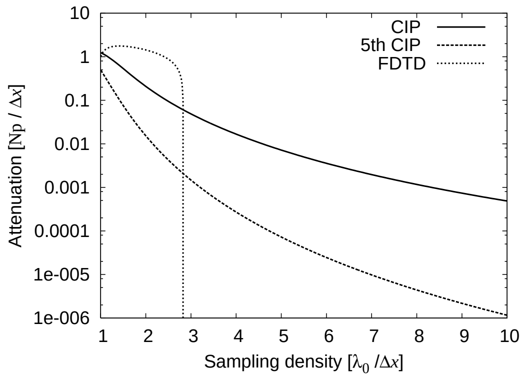

Figure 4: Numerical attenuation characteristics of CIP and FDTD.

3 Conclusion

We have derived the numerical dispersion relation for the CIP methods of the third and the fifth order. The derived numerical dispersions can predict successfully the waveforms computed by the CIP methods. The numerical phase velocity and the numerical attenuation are compared with the ones of FDTD method. In the case of the phase velocity, the CIP methods show very good performance, while it is found that the CIP methods have the numerical attenuation, and it is not negligible especially for the case of 3rd CIP.

References

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A The coefficients for the updating equations of the 5-th order CIP

The coefficients appearing in the Eqs.(12)-(14) are given by

$$A'_1 = (1-\xi)^3(1+3\xi+6\xi^2), \quad A'_2 = \xi^3(10-15\xi+6\xi^2), \quad A'_3 = -\xi\Delta x(1-\xi)^3(1+3\xi),$$
$$A'_4 = \xi^3\Delta x(1-\xi)(4-3\xi), \quad A'_5 = \frac{\xi^2\Delta x^2}{2}(1-\xi)^3, \quad A'_6 = \frac{\xi^3\Delta x^2}{2}(1-\xi)^2,$$
$$B'_1 = \frac{30\xi^2}{\Delta x}(1-\xi)^2, \quad B'_2 = -B'_1, \quad B'_3 = (1-\xi)^2(1-3\xi)(1+5\xi),$$
$$B'_4 = -\xi^2(2-3\xi)(6-5\xi), \quad B'_5 = -\frac{\xi\Delta x}{2}(1-\xi)^2(2-5\xi), \quad B'_6 = -\frac{\xi^2\Delta x}{2}(1-\xi)(3-5\xi),$$
$$C'_1 = -\frac{60\xi}{\Delta x^2}(1-\xi)(1-2\xi), \quad C'_2 = -C'_1, \quad C'_3 = \frac{12\xi}{\Delta x}(1-\xi)(3-5\xi),$$
$$C'_4 = \frac{12\xi}{\Delta x}(1-\xi)(2-5\xi), \quad C'_5 = (1-\xi)(1-8\xi+10\xi^2), \quad C'_6 = \xi(3-12\xi+10\xi^2).$$

B The dispersion relation for CIP of the 5-th order

The explicit equation of the numerical dispersion relation of CIP method of the 5-th oder is given in the following equation.

$$\sum_{m=0,n=0}^{3} D'_{mn}\exp\{j(m\tilde{k}_n + n\xi)P\} = 0, \tag{19}$$

where D'_{mn} are given by

$$D_{30} = \xi^9, \quad D_{21} = -3\xi^4(2-10\xi+16\xi^2-8\xi^3+\xi^4), \quad D_{20} = 3\xi^4(1-\xi)(2+2\xi-2\xi^2-2\xi^3+\xi^4),$$
$$D_{12} = 3\xi(1-8\xi+16\xi^2-10\xi^3+2\xi^4), \quad D_{11} = -6\xi(1-\xi)(1-3\xi-11\xi^2+27\xi^3-11\xi^4-3\xi^5+\xi^6),$$
$$D_{10} = 3\xi(1-\xi)^4(1+4\xi-2\xi^2-2\xi^3+\xi^4), \quad D_{03} = -1, \quad D_{02} = 3(1-\xi)(1-2\xi-2\xi^2+2\xi^3+2\xi^4),$$
$$D_{01} = -3(1-\xi)^4(1-2\xi-2\xi^2+4\xi^3+\xi^4), \quad D_{00} = (1-\xi)^9,$$

and for the other combinations of m and n, $D_{mn} = 0$.