POS1-30

About One Semi-analytic Method for the Solution of Scattering Problems with 3D Dielectric Scatterers

[#]Aleksey Serebrennikov [1]

[1] Department of Physics, Perm Federal Technical University
614990, Russian Federation, Perm, Komsomolsky Avenue,29

1. Introduction

In this paper a certain integrodifferential equations are proposed for the solution of EM problems with 3D dielectric bounded scatterers. This article can be viewed as an extension of our previous work [1]. It is also necessary to point out that Waterman's [2] and Kyurkchan's [3] methods serve as a background of this work.

2. Differential Formulation

Suppose that certain EM field with strength $\vec{E}_0, \vec{H}_0$ and time dependence $e^{-i\omega t}$ propagates in linear isotropic uniform medium. Let the scatterer bounded with the smooth surface S occupies the finite volume of space $\overline{G}_2 = G_2 \cup S$. Complement of $\overline{G}_2$ to entire space R^3 we denote as $G_1 = R^3 / \overline{G}_2$ ($\overline{G}_1 = G_1 \cup S$). Interaction of the incident field with an inhomogeneity results in the appearance of scattered and interior fields which we denote as $\vec{E}_1, \vec{H}_1$ and $\vec{E}_2, \vec{H}_2$, respectively. We shall refer to the properties of the surrounding medium by utilizing script "1" and script "2" will be used for denoting the constants of the material filling the body. By means of Maxwell's equations and classical constitutive relations one can show that the fields $\vec{E}_j, \vec{H}_j$ ($j=1,2$) satisfy the uniform Helmholtz equations. On the surface S the tangential parts of field vectors must be equated:

$$\vec{n} \times (\vec{E}_0 + \vec{E}_1)\big|_S = \vec{n} \times \vec{E}_2\big|_S, \quad \vec{n} \times (\vec{H}_0 + \vec{H}_1)\big|_S = \vec{n} \times \vec{H}_2\big|_S. \tag{1}$$

The denotation $\vec{n}$ corresponds to the unit normal to S pointing outward the volume of the scatterer $\overline{G}_2$. We shall also use the denotation $\hat{n}$ that corresponds to the normal pointing inside the body. Additionally, the vectors $\vec{E}_1, \vec{H}_1$ must satisfy the Silver – Muller radiation conditions [2].

3. Stratton-Chu Integrals for Interior and Exterior Field

The starting point to yield Stratton-Chu representations is the integration of Helmholtz equations over the volume $\overline{G}_j$ in the following way: $\int\limits_{\overline{G}_j} (\vec{\nabla} \cdot \vec{\nabla} \vec{F}_j) \cdot (\vec{a} g_j) \, dV + \int\limits_{\overline{G}_j} k_j^2 \vec{F}_j \cdot (\vec{a} g_j) \, dV = 0$,

where $\vec{F}_j \equiv \vec{E}_j, \vec{H}_j$; $j=1,2$; $\vec{a}$ is the arbitrary constant vector, $g_j = \exp(ik_j|\vec{r} - \vec{r}'|)/4\pi|\vec{r} - \vec{r}'|$ denotes free space Green's function. By utilizing several identities of vector analysis and Gauss theorem one may define the following integral representations:

$$\begin{aligned}
\int\limits_{\overline{G}_2} \vec{E}_2 \, \delta(|\vec{r} - \vec{r}'|) \, dV &= -\int\limits_S \left\{ (\vec{n} \cdot \vec{E}_2)\vec{\nabla} + (\vec{n} \times \vec{E}_2) \times \vec{\nabla} + ik_2 Z_2 (\vec{n} \times \vec{H}_2) \right\} g_2 \, dS, \\
\int\limits_{\overline{G}_2} \vec{H}_2 \, \delta(|\vec{r} - \vec{r}'|) \, dV &= -\int\limits_S \left\{ (\vec{n} \cdot \vec{H}_2)\vec{\nabla} + (\vec{n} \times \vec{H}_2) \times \vec{\nabla} - i(k_2/Z_2)(\vec{n} \times \vec{E}_2) \right\} g_2 \, dS,
\end{aligned} \tag{2}$$

where $\delta(|\vec{r} - \vec{r}'|)$ denotes Dirac delta. Evidently, the exact form of the l.h.s. in (2) depends on the concrete location of the observation point $\vec{r}'$ in space. Two situations are possible: if $\vec{r}' \in \overline{G}_2$ then

the integral on the l.h.s. presents electric field at location $\vec{r}'$, otherwise ($\vec{r}' \in G_I$) it will be equal to zero. In relation (2) we may carry differential operators of observer to source points as well as take curl operator of both sides of the equality. By utilizing the identities $\vec{e}'_r \cdot \vec{\nabla}' \times (\vec{j} g_2) \equiv r'^{-1} \vec{j} \cdot \vec{\nabla} \times (\vec{r} g_2)$ and $\vec{e}'_r \cdot \vec{\nabla}' \times \vec{\nabla}' \times (\vec{j} g_2) \equiv r'^{-1} \vec{j} \cdot \vec{\nabla} \times \vec{\nabla} \times (\vec{r} g_2)$ [4] one finds the expression for the radial components of EM vectors (magnetic component is only shown for the sake of brevity):

$$\int_{\bar{G}_2} H_{2r} \delta(|\vec{r} - \vec{r}'|) dV = \frac{i}{k_2 Z_2} \frac{1}{r'} \int_S (\vec{n} \times \vec{E}_2) \cdot \vec{K}_2[g_2] dS - \frac{1}{r'} \int_S (\vec{n} \times \vec{H}_2) \cdot \vec{K}_I[g_2] dS, \tag{3}$$

where $\vec{K}_1[\cdot] \equiv \vec{\nabla} \times (\vec{r} \cdot)$, $\vec{K}_2[\cdot] \equiv \vec{\nabla} \times \vec{K}_I[\cdot]$. Analogous expressions of field vectors we may get for exterior space:

$$\int_{\bar{G}_I} H_{Ir} \delta(|\vec{r} - \vec{r}'|) dV = \frac{i}{k_I Z_I} \frac{1}{r'} \int_S (\hat{n} \times (\vec{E}_0 + \vec{E}_I)) \cdot \vec{K}_2[g_I] dS - \frac{1}{r'} \int_S (\hat{n} \times (\vec{H}_0 + \vec{H}_I)) \cdot \vec{K}_I[g_I] dS. \tag{4}$$

4. Integrodifferential System of Equations and Algebraic Problem

If we desire to yield constitutive integrodifferential equations by means of (3) and (4) that allow obtaining unknown EM fields we need suitable operator forms of vectors $\vec{E}_j, \vec{H}_j$. For free space we choose the following differential relations:

$$\vec{E}_I[u, v] = \vec{K}_2[v] + ik_I Z_I \vec{K}_I[u], \quad \vec{H}_I[u, v] = \vec{K}_2[u] - i(k_I/Z_I) \vec{K}_I[v], \tag{5}$$

where scalar fields u and v are expressed by the Atkinson-Wilcox series [2]. These representations satisfy "uniform" Maxwell's equations (free of volume currents on the r.h.s.) as well as Silver-Muller radiation conditions.

Before we choose the operator form of interior field let us speak out the following consideration. Physics of the solution requires that analytical representation being used has to be free of singularities inside the scatterer. In simplest case, we can get this result by using vector spherical harmonics expansion which radial multipliers contain the Bessel function of the first kind j_n. Therefore, in analogy with (5) one yields:

$$\vec{E}_2[a, b] = \vec{K}_2[b] + ik_2 Z_2 \vec{K}_I[a], \quad \vec{H}_2[a, b] = \vec{K}_2[a] - i(k_2/Z_2) \vec{K}_I[b], \tag{6}$$

thereby a and b functions are expressed by means of the series:

$$\begin{Bmatrix} a(\vec{r}) \\ b(\vec{r}) \end{Bmatrix} = \sum_{nm} \begin{Bmatrix} a_{nm} \\ b_{nm} \end{Bmatrix} (-i)^{n+1} j_n(k_2 r) Y_{nm}(\theta, \varphi),$$

where $Y_{nm}(\theta, \varphi) = \bar{P}_n^{|m|}(\cos\theta) e^{im\varphi}$ - spherical functions. Consider equalities (3) and (4). As a first step we perform the "exchange" by boundary quantities of EM field in the integrand terms according to boundary conditions (2) [3,5]. After that we put the observation point $\vec{r}'$ in far zone. Transform first equation in (4). It is evident that in this case we have to choose the nonzero magnitude of the integral on the l.h.s. since the point $\vec{r}'$ is placed in G_I. Furthermore, in far zone we yield asymptotic forms of Green's function and Atkinson-Wilcox series for the radial component of the magnetic vector H_{Ir} as follows [1]: $g_I = \frac{e^{ik_I r'}}{4\pi r'} f_I + O(r'^{-2})$, $H_{Ir} = -\frac{e^{ik_I r'}}{r'} \frac{B u_0(\theta', \varphi')}{k_I r'} + O(r'^{-3})$, where

$f_I \equiv e^{-i\vec{k}_I \cdot \vec{r}}$ and B denotes Beltrami operator [2]. By substituting latter expressions in (4) we come to the integrodifferential equation:

$$B u_0 = \frac{i}{4\pi Z_I} \int_S (\vec{n} \times \vec{E}_2) \cdot \vec{K}_2[f_I] dS - \frac{k_I}{4\pi} \int_S (\vec{n} \times \vec{H}_2) \cdot \vec{K}_I[f_I] dS. \tag{7}$$

Now we consider integral (3) more explicitly. Since observation point we hold in G_I we need to take the zero magnitude of the integral on the l.h.s. in (3) [5]. By utilizing the asymptotic form of Green's function and rendering identical transforms one yields the equation [5]:

$$0 = \frac{i}{4\pi Z_2} \int_S (\vec{n} \times (\vec{E}_0 + \vec{E}_I)) \cdot \vec{K}_2[f_2] dS - \frac{k_2}{4\pi} \int_S (\vec{n} \times (\vec{H}_0 + \vec{H}_I)) \cdot \vec{K}_I[f_2] dS, \tag{8}$$

where $f_2 \equiv e^{-i\vec{k}_2 \cdot \vec{r}}$. Another one pair of integrodifferential equations we may define by the use of Stratton-Chu integrals expressing electric vectors. Taking into account the construction of EM field operators in the integrand terms one can deduce that equations (7)-(8) present integrodifferential system with respect to the unknown functions $u_0(\theta', \varphi')$, $v_0(\theta', \varphi')$ [1] and algebraic coefficients a_{nm} and b_{nm} as well. Analysis of the structure of equations (7) - (8) leads us to the following inference: equation (7) presents generalization of the equation defined in [1] and another equation is intrinsic to the null-field method [2].

Finally integrodifferential system (7)-(8) can be reduced to pure algebraic one by expanding u_0 and v_0 functions into Fourier series with respect to spherical harmonics basis as it has although been done in [1]. Thus we get the algebraic analog of (7) – (8) as:

$$\vec{w}_1 = \tilde{M}_2 \vec{w}_2, \quad \tilde{M}_1 \vec{w}_1 = -\vec{V}. \tag{9}$$

It should be mentioned that the first equation in (9) coincides with the same in [3]. A few words have to be said regarding the solution algorithm that results from the construction of algebraic system (9). The second equation in (9) does not contain the vector of unknowns $\vec{w}_2$, hence, these two linear systems can be solved consequently. In fact, in many cases the definition of interior field is not obligatory. For instance, in problems with dielectric resonator antennas designs, the quantities characterizing their efficiency (radiation patterns, spectral responses) can be found via exterior field. Thus, the computational complexity of the algorithm differs slightly compared to a perfectly conducting body problem [1].

5. Validation

The aim which we pursue in this section is to verify equations (7)-(8) by the solution of well-known canonical problems. We begin with the analytical solution for dielectric sphere excited by plane wave. Thus, let linearly polarized plane wave

$$\vec{E}_0 = E_0 \, \vec{e}_x \, e^{i\vec{k}_1 \cdot \vec{r}}, \quad \vec{H}_0 = (E_0/Z_1) \vec{e}_y \, e^{i\vec{k}_1 \cdot \vec{r}} \tag{10}$$

propagates in free space along the z axis in positive direction. Let also dielectric sphere of radius R is placed in space so that its center coincides with the origins of Cartesian and spherical coordinate systems. By utilizing known identities of spherical analysis [6] it is easy to show that the equations (9) result in the form:

$$\left(\frac{ik_2 Z_2}{Z_1} \tilde{M}_{12}^{F_2 F_1} + ik_1 \tilde{M}_{21}^{F_2 F_1} \right) \vec{a} = \vec{u}, \quad \left(\frac{ik_1 Z_1}{Z_2} \tilde{M}_{12}^{T_1 F_2} + ik_2 \tilde{M}_{21}^{T_1 F_2} \right) \vec{u} = -\vec{V}^u,$$

$$\left(\frac{ik_2 Z_1}{Z_2} \tilde{M}_{12}^{F_2 F_1} + ik_1 \tilde{M}_{21}^{F_2 F_1} \right) \vec{b} = \vec{v}, \quad \left(\frac{ik_1 Z_2}{Z_1} \tilde{M}_{12}^{T_1 F_2} + ik_2 \tilde{M}_{21}^{T_1 F_2} \right) \vec{v} = -\vec{V}^v, \tag{11}$$

The elements of $\tilde{M}_{pq}^{F_j T_l}$ are written as $M_{pq\,nm\,ab}^{F_j T_l} = \frac{1}{2\pi n(n+1)} \int\limits_S (\vec{n} \times \vec{K}_p [F_{ab}(k_j, \vec{r})]) \cdot \vec{K}_q [T_{nm}^*(k_l, \vec{r})] dS$,

$M_{pq\,nm\,ab}^{F_j F_l} = \frac{1}{2\pi n(n+1)} \int\limits_S (\vec{n} \times \vec{K}_p [F_{ab}(k_j, \vec{r})]) \cdot \vec{K}_q [F_{nm}^*(k_l, \vec{r})] dS$, where $T_{nm}^*(k, \vec{r}) = i^{n+1} h_n^{(1)}(kr) Y_{nm}^*(\theta, \varphi)$,

$F_{nm}^*(k, \vec{r}) = (-i)^{n+1} j_n(kr) Y_{nm}^*(\theta, \varphi)$. The elements of the vectors $\vec{V}^u = \{V_{nm}^u\}$, $\vec{V}^v = \{V_{nm}^v\}$ have the form

$$V_{nm}^u = \frac{1}{2\pi n(n+1)} \left\{ \frac{1}{Z_2} \int\limits_S (\vec{n} \times \vec{E}_0) \cdot \vec{K}_2 [F_{nm}^*(k_2, \vec{r})] \, dS + ik_2 \int\limits_S (\vec{n} \times \vec{H}_0) \cdot \vec{K}_1 [F_{nm}^*(k_2, \vec{r})] \, dS \right\},$$

$$V_{nm}^v = \frac{1}{2\pi n(n+1)} \left\{ -Z_2 \int\limits_S (\vec{n} \times \vec{H}_0) \cdot \vec{K}_2 [F_{nm}^*(k_2, \vec{r})] \, dS + ik_2 \int\limits_S (\vec{n} \times \vec{E}_0) \cdot \vec{K}_1 [F_{nm}^*(k_2, \vec{r})] \, dS \right\}.$$

By utilizing the system of integral identities from [6] it can be easily proved that the system (11) results in the well-known Mie solution [6].

Let us go in for searching the numerical solutions. In our investigation the diffraction of normally incident plane waves on dielectric cubes of different size and permittivity has been considered. The results have been compared against known literature data whenever possible. The 2d and 3d RCS images will be shown during presentation. Herein we study the convergence of the algorithm and its time expenses for the following example. Consider diffraction of plane wave on a cube having *0.5* meter on a side. In this instance we shall keep frequency unchanged (*f≈300* MHz),

only refractive index n will be altered within a range 3÷8. The criterion of convergence will be a minimum of relative error: $\delta_N = |P_N - P_{N-1}| / \min(P_{N-1}, P_N)$, where P_N denotes radiated power [1]. Subscript N shows the highest order of spherical harmonics which we hold in the truncated Fourier series. The results are accumulated in Table 1. The columns CPU1 and CPU2 denote the time of algebraic system forming and solution, respectively, for different orders N and magnitudes of refractive index. As seen, in all the cases system forming time greatly exceeds a solution time, though we used most time expensive solver based on Gaussian elimination. Note that algebraic system forming time is mainly determined by the calculation time of r.h.s. and l.h.s. integrals in (9). This time, obviously, depends on two issues: the number of harmonics which we hold in the truncated series and the discretization order of cube surface used in numerical integration. The discretization order has been found separately for each case shown in Table 2 in numerical tests; the cube surface has been subdivided into square patches. The second order quadrature formula (four nodes per patch) has been applied within a patch. The discretization order as a function of refractive index is shown in Table II, where the parameter N_{pe} denotes the number of patch elements along arbitrary edge notwithstanding its spatial orientation. The data shown in the tables result in the inference: no problems have been observed with the objects having high contrast dielectric filling with respect to the environment.

Table 1: Relative errors and time of computation

Order N / δ_N , % / CPU1, sec / CPU2, sec			
$n = 4.0$	$n = 5.0$	$n = 6.0$	$n = 7.0$
10/16.9/33./1.	13/.7/125./4.	16/9.2/376./11.	17/15.7/565./16.
11/1./47./1.6	14/2.95/164./6.	17/1.6/474./16.	18/10.2/700./22.
12/.89/65./2.5	15/.52/212./8.	18/1./585./22.	19/.38/857./29.
13/.06/88./3.8	16/.43/266./11.	19/.25/703./29.	20/.04/1039./39.

Table 2: Discretization order versus refractive index

N_{pe}	n		N_{pe}	n		N_{pe}	n
10	3.0		12	5.0		16	7.0
10	4.0		14	6.0		18	8.0

References

[1] A. M. Serebrennikov, "Boundary integrodifferential equations for electromagnetic scattering problems in three dimensions," IEEE Trans. Antennas Propagat., vol.54, No.8, pp.2337-2344.

[2] D. Colton and R. Kress, *Integral Equation Methods in Scattering Theory*. New York: Wiley, 1983.

[3] A. G. Kyurkchan, D. B. Demin, "Electromagnetic Wave Diffraction from Magnetodielectric and Absorbing Scatterers," J. Comm. Tech. and Electronics, vol.49, No.11, pp.1218-1227, 2004.

[4] G. T. Markov and A. F. Chaplin, *Excitation of Electromagnetic Waves (in Russian)*. Moscow: Radio and Comm., 1983.

[5] A. M. Serebrennikov, "Boundary integrodifferential equations for the solution of electromagnetic scattering problems for metal-dielectric bodies," Proc. of 16th Int. Symp. EMC 2005, Zurich, Switzerland, pp. 429-434, 13-18 Feb. 2005.

[6] C. F. Bohren, D. R. Huffman, *Absorption and Scattering of Light by Small Particles.* New York: John Wiley, 1983.