Decoding Algorithms with Reduced Computational Requirements for Iterative Receivers

Michal Sybis and Piotr Tyczka
Chair of Wireless Communications, Poznan University of Technology
ul. Polanka 3, 60-965 Poznan, Poland
Email: michal.sybis@put.poznan.pl, tyczka@et.put.poznan.pl

***Abstract*—Iterative processing, originated from the introduction of turbo codes, has become prevalent technique in modern receiver design. Its applications extend beyond channel decoding and include signal detection, channel estimation and equalization, interference suppression, and synchronization. In this paper, in the aspect of energy efficiency in wireless communication systems, we present and compare several reduced complexity algorithms for component decoding modules in iterative receivers. The comparison is carried out in terms of computational requirements and error performance for the case of turbo TCM systems.**

I. Introduction

The advent of parallel concatenated convolutional (turbo) codes with the iterative decoding technique [1], [2] is undoubtedly one of the most significant breakthroughs in modern communications in the last decades. This high recognition of turbo codes stems from the fact that they offer performance approaching the Shannon capacity limit with the decoding procedure of reasonable computational complexity. The turbo decoding principle based on the idea of exchanging information between component decoders in an iterative manner has also been adapted to other code classes and transmission techniques, including schemes that have emerged as extensions of a turbo coding concept (often referred to as *turbo-like codes*). Examples of systems employing iterative decoding in the receiver include serially concatenated codes [3], low-density parity-check codes (LDPC) [4], bit-interleaved coded modulation (BICM) [5], turbo trellis-coded modulation (TTCM) [6] and space-time turbo codes (STTC) [7]. Furthermore, the idea of passing information back and forth between different components in a receiver (so-called *iterative processing* or *turbo processing*) has found applications not only in channel decoding but also in signal detection, channel estimation and equalization, interference suppression, and synchronization, becoming prevalent in state-of-the-art receiver design [8].

Iterative decoding makes use of soft-input soft output (SISO) component decoders. The optimal algorithm for SISO decoders is symbol-by-symbol maximum *a posteriori* probability (MAP) algorithm [9] which is practically implemented in the log-domain and known as the Log-MAP algorithm [10]. The calculation of the logarithm of the sum of exponentials, denoted as $\max^*$ operator [11], constitutes a significant, and usually dominant, portion of the overall decoder complexity. The manner that $\max^*$ is implemented is critical to the performance and complexity of the decoder. Due to this, reduced complexity decoding algorithms for SISO modules have received a considerable amount of attention in recent years. All the approaches that have appeared in the literature target at a simplification of the $\max^*$ computations without a substantial loss of the decoding performance.

The use of simplified decoding algorithms in the receiver is also favorable in terms of the energy efficiency, particularly in the wireless transceivers. Reducing the number of operations performed by the digital signal processor (DSP) per decoding step translates to energy savings in the wireless device that is highly desirable feature. In this paper, we deal with reduced complexity algorithms for SISO decoders in wireless iterative receivers in the context of energy efficiency. In particular, we review the algorithms recently proposed in [12]-[16] in terms of the computational requirements and resulting energy savings for the case of TTCM systems. Simulated bit error rate (BER) performance results for AWGN and uncorrelated Rayleigh fading channels are also given. The remainder of this paper is organized as follows. Section II describes the iterative receiver structure and operation principle, and Section III presents the investigated reduced-complexity algorithms. Complexity comparison of the algorithms is provided and discussed in Section IV. Simulation results are given in Section V, and Section VI contains concluding remarks.

II. Iterative Receiver

The block diagram of a generic iterative receiver is depicted in Fig. 1. Depending on the system, as the SISO modules operate appropriate blocks, e.g., component decoders for turbo codes or parallel TTCM, inner decoder and outer decoder for serially concatenated codes or serial TTCM, demodulator and decoder for BICM, etc. Hereafter in this Section, we assume the structure of an iterative decoder for parallel concatenated codes in Fig. 1. The general idea of iterative decoding is to provide some 'soft' information about the the decoded symbols from the output of one component decoder to the input of the other in an iterative manner in order to improve decisions about the data symbols. The presence of a feedback path allows to realize successive iterations. Soft information generated and accepted by SISO decoders has a character of reliability information and usually takes the form of a *log-likelihood ratio* (LLR) for each data symbol. As shown in Fig. 1, the inputs to each decoder are the received (*intrinsic*) information and the *extrinsic* information. The former comes from the received data symbol whereas the extrinsic information is produced by the other decoder and reflects its *beliefs* regarding the data, achieved after completion decoding in current iteration. The extrinsic information is used as *a priori* information for the SISO decoder in the next decoding step. The output of each decoder contains only extrinsic information to pass on

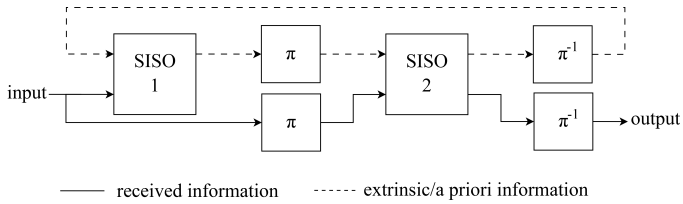

Fig. 1. Block diagram of a generic iterative receiver (π – interleaver, π^{-1} – deinterleaver).

to the next decoder. After performing decoding by both SISO decoders in sequence, the whole process iterates again. Once the iterations have been completed, a hard decision on data symbols is taken using the output LLRs from the SISO 2 component decoder. For the TTCM codes, investigated in this paper, the iterative receiver operates essentially in a fashion described above.

III. Simplified Decoding Algorithms for SISO Decoders

In the Log-MAP algorithm, the calculation of the soft output as well as the forward and backward metrics, and the branch metrics of trellis transitions requires computation of the $\max^*$ operator defined as

$$\max{}^*(x_1, \ldots, x_n) = \ln\left(\sum_{i=1}^{n} e^{x_i}\right) \tag{1}$$

An exact solution to this problem, used in the Log-MAP algorithm, is the application of the Jacobian logarithm

$$\begin{aligned}\max{}^*(x_1, x_2) &= \max(x_1, x_2) + \ln(1 + e^{-|x_2 - x_1|}) \\ &= \max(x_1, x_2) + f_c(|x_2 - x_1|)\end{aligned} \tag{2}$$

where $f_c(.)$ is a correction function. To obtain the $\max^*$ operator for more than two arguments, i.e. $n > 2$, the Jacobian logarithm (2) is applied recursively $n - 1$ times. For example, assuming $n = 3$ it yields:

$$\max{}^*(x_1, x_2, x_3) = \max{}^*(\max{}^*(x_1, x_2), x_3). \tag{3}$$

In order to minimize the complexity of the Log-MAP algorithm, the correction function $f_c(.)$ is practically implemented with a look-up table (LUT) with eight values [10]. If the correcting value of the LUT is omitted, then the Log-MAP algorithm simplifies to the Max-Log-MAP algorithm [10].

Since the Jacobian logarithm must be applied in a recursive manner of (3) for $n > 2$ arguments, it is intuitive to expect that a $\max^*$ approximation with n arguments may bring complexity reductions as compared with the conventional Jacobian logarithm solution. Motivated by this reasoning, the following approximations for the $\max^*$ operator with n arguments have been recently proposed.

First algorithm, referred to as AvN Log-MAP [16], is derived from two inequalities. The first origins from the definition of the $\max^*$ operator given in (1), while for the second one, the Jensen inequality (4) is considered:

$$\frac{\sum_{i=1}^{n} \alpha_i}{n} \geq \left(\prod_{i=1}^{n} \alpha_i\right)^{\frac{1}{n}}, \ \alpha_i > 0. \tag{4}$$

Based on these two inequalities, the new approximation for the $\max^*$ is obtained and expected to have a better performance than the Max-Log-MAP algorithm. The approximation is formulated as

$$\max{}^*(x_1, \ldots, x_n) \approx \max\left(\max_{i=1:n}(x_i), \frac{1}{N}\sum_{i=1}^{n} x_i\right), \tag{5}$$

where N is the parameter of the approximation. For a given transmission scheme, an optimal value of N minimizing BER performance at the assumed signal-to-noise ratio (SNR) level can be found by means of computer simulations.

Reduced complexity algorithm, denoted as LM-n, was proposed in [15]. This approach is derived from the Chebyshev inequality

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \leq n\sum_{i=1}^{n} a_i b_i, \tag{6}$$

where $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n \in \Re$. The final formula for the LM-n algorithm is the following

$$\max{}^*(x_1, \ldots, x_n) \approx \max_{i=1:n}(x_i) + \ln\left(1 + \frac{(n-1)}{n} e^{-|z|}\right), \tag{7}$$

where z is the absolute value of the difference between the maximum value and the second maximum value among n arguments of $\max^*$ operator. As seen, the first term of the approximation is a simple max operation. The second term of (7) can be thought as a correction function $f_c(.)$. For the given n, the curve of the correction function can be analyzed and approximated with several values stored in small LUT. In our implementation, we have approximated the correction function with eight values.

Another algorithm that is also based on the Chebyshev inequality was introduced in [14]. The LM-n-q algorithm uses the following approximation for the $\max^*$ operator with n arguments:

$$\max{}^*(x_1, \ldots, x_n) \approx \max_{i=1:n}(x_i) + \ln\left(1 + q \cdot e^{-|z|}\right), \tag{8}$$

where, similar as in the LM-n algorithm, z is the absolute difference between the maximum value and the second maximum value among n arguments of $\max^*$ operator, and q is an additional parameter that has influence on the correcting term. For a given transmission scheme, the optimal value of the parameter q that minimizes BER at the assumed signal-to-noise ratio (SNR) level can been found through computer simulations. Comparing approximations (7) and (8), it is easily seen that the LM-n and LM-n-q algorithms differ only in the form of the correction function $f_c(.)$. As in the LM-n implementation, eight values of the correction function were stored in LUT used in our simulations.

Other solution for equation (2) is presented in [12], [13]. Instead of approximating $f_c(.)$ and for pure mathematical purposes, a novel approximation of the Jacobian logarithm has been obtained by solving a geometric programming problem in [17]. The authors in [17] have approximated (2) as a whole, i.e. the $\max^*$ operator directly. In particular, the Jacobian

logarithm has been simplified into r piecewise-linear (PWL) approximation terms deploying the max operation

$$\max^*(x_1, x_2) \approx \max(\kappa_1 x_1 + \lambda_1 x_2 + \mu_1, \ldots, \kappa_i x_1 + \lambda_i x_2 + \mu_i), \quad (9)$$

where κ_i, λ_i and μ_i are real positive values and $i \geq 2$. The best PWL approximations of the $\max^*$ operator with different number of terms, denoted with r, can be found in [17]. In general, the approximation error reduces in the order of $\frac{\sqrt{2}}{r^2}$ and for practical applications $2 \leq r \leq 5$ has been considered.

A relatively simple expression for the $\max^*$ approximation is obtained for $r = 3$ as

$$\max^*(x_1, x_2) \approx \max(x_1, 0.5 \cdot x_1 + 0.5 \cdot x_2 + 0.693, x_2). \quad (10)$$

The least square approximation error resulting from (10) as compared with the exact computation from (2) is equal to 0.223. The Log-MAP decoding algorithm using the PWL approximation with $r = 3$ terms will be examined in Sections IV and V. It can also be mentioned that for the turbo decoding, the $r = 2$ approximation is identical to the Max-Log-MAP algorithm.

IV. Computational Requirements Comparison

From the implementation point of view, and also the potential capabilities of energy savings, the key aspect of the algorithms is their computational requirements. The reduced complexity algorithms presented in Section III have been compared against the optimal Log-MAP and the simple Max-Log-MAP algorithms for the instance of turbo TCM transmission. We have considered both parallel and serial concatenated TTCM schemes [6], [18]. The parallel concatenated TTCM scheme has employed two systematic feedback rate-3/4 8-state TCM encoders with parity-check coefficients (in octal form): $\mathbf{h}^{(0)}$=11, $\mathbf{h}^{(1)}$=02, $\mathbf{h}^{(2)}$=04, $\mathbf{h}^{(3)}$=10, and 16-QAM modulation. In the serial concatenated TTCM scheme, a rate-2/3 convolutional encoder with $\mathbf{h}^{(0)}$=13, $\mathbf{h}^{(1)}$=15, $\mathbf{h}^{(2)}$=17, was applied as an outer encoder. As an inner encoder, we used the same encoder as in the parallel TTCM scheme. Hence, the overall code rate of the serial TTCM scheme is $R_c = 1/2$. Complexity comparison of the algorithms has been performed for software (i.e., computer based) implementation of the TTCM systems.

Tables I and II depict the required number of operations (i.e., additions, multiplications, comparisons, bit shifts, conversion to integer and assignment) per single decoding step of the algorithms for the parallel and serial concatenated TTCM scheme, respectively. The following notation is used in the tables (and in the figures of Section V): LM-AvN – AvN Log-MAP, LM-r – PWL approximation with $r = 3$ terms, LM – Log-MAP, MLM – Max-Log-MAP. As shown in Tables I and II, among the new algorithms the AvN Log-MAP algorithm has the lowest computational requirements in both TTCM scenarios. Its reduction in the number of operations with respect to the Log-MAP is significant and amounts to 41.1% in parallel and 31.2% in serial TTCM scheme. When compared with the Max-Log-MAP algorithm, it is found that the AvN Log-MAP algorithm requires 34.0% and 43.3% more operations, respectively. One can also easily notice that the LM-n and LM-n-q algorithms have the same complexity. This result was expected since, as it was mentioned in Section III, both algorithms differ only in the form of the correction function $f_c(.)$.

TABLE I. Required number of operations for decoding algorithms per single decoding step for parallel TTCM scheme

Algorithm	LM	LM-AvN	LM-n	LM-n-q	LM-r	MLM
Additions	680	512	392	392	680	344
Multiplications	0	24	24	24	0	0
Comparisons	357	213	357	357	357	189
Bit shifts	168	0	0	0	168	0
Conversion to int	168	0	24	24	0	0
Assignment	233	197	269	269	281	173
OVERALL	1606	946	1066	1066	1486	706

TABLE II. Required number of operations for decoding algorithms per single decoding step for serial TTCM scheme

Algorithm	LM	LM-AvN	LM-n	LM-n-q	LM-r	MLM
Additions	492	432	312	312	492	264
Multiplications	0	24	24	24	0	0
Comparisons	249	165	261	261	249	141
Bit shifts	114	0	0	0	114	0
Conversion to int	114	0	24	24	0	0
Assignment	185	173	221	221	233	149
OVERALL	1154	794	842	842	1088	554

Tables III and IV summarize the comparison of the algorithms in terms of overall number of operations in the parallel and serial scenarios, respectively. It can be easily seen that the reduction in complexity of the LM-n and LM-n-q algorithms against the Log-MAP is also substantial. Both algorithms are in parallel TTCM scheme 33.6%, and in serial scheme 27%, simpler than the Log-MAP algorithm. The PWL approximation with $r = 3$ terms offers relatively modest reduction in the number of operations, i.e., 7.5% and 5.7%, respectively.

TABLE III. Complexity comparison of decoding algorithms per single decoding step for parallel TTCM scheme – overall number of operations

Algorithm	Overall number of operations	Reduction wrt LM
LM	1606	—
LM-AvN	946	41.1%
LM-n	1066	33.6%
LM-n-q	1066	33.6%
LM-r	1486	7.5%
MLM	706	56.0%

TABLE IV. Complexity comparison of decoding algorithms per single decoding step for serial TTCM scheme – overall number of operations

Algorithm	Overall number of operations	Reduction wrt LM
LM	1154	—
LM-AvN	794	31.2%
LM-n	842	27.0%
LM-n-q	842	27.0%
LM-r	1088	5.7%
MLM	554	52.0%

From the analysis given in this section, we conclude that the AvN Log-MAP, the LM-n and the LM-n-q algorithms offer significant savings in decoding effort with respect to the Log-MAP. This result makes them attractive for implementation in iterative receivers and implies improved energy efficiency. It should also be emphasized that all complexity comparisons presented in Tables I–IV are per single decoding step in a constituent SISO decoder and hence, these results do not depend on the number of iterations or the block size of the TTCM scheme.

V. Error Performance Comparison

In order to evaluate performance of the simplified algorithms, computer simulations were run for parallel and serial concatenated TTCM schemes from Section IV. In simulations, the AWGN channel and the uncorrelated, i.e., fully interleaved, Rayleigh fading channel with perfect channel state information (CSI) were considered. The block sizes of $K = 684$ and $K = 5000$ symbols with the S-random interleavers with the spreading factors $S = 7$ and $S = 13$, respectively, were assumed. For comparison purposes, the BER performance curves for the conventional Log-MAP (with a LUT storing eight values) and Max-Log-MAP algorithms were also evaluated. At the receiver, 8 decoding iterations for all algorithms were performed.

BER performance evaluation results in the AWGN channel for the parallel TTCM and both small ($K = 684$ symbols) and large ($K = 5000$ symbols) interleaver sizes are given in Figs. 2 and 3, respectively. As it can be seen, the LM-n-q algorithm practically has the Log-MAP performance. It outperforms the LM-n and LM-r algorithms by 0.1 dB and the AvN Log-MAP by 0.2–0.3 dB at BER = 10^{-4}. In turn, the AvN Log-MAP algorithm is superior to the Max-Log-MAP by 0.1–0.2 dB at the same BER level.

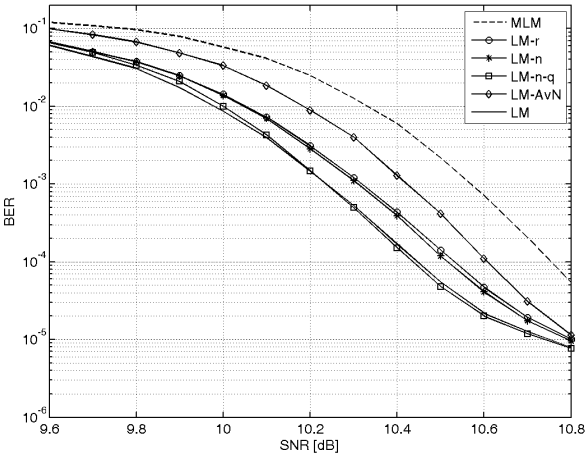

Fig. 2. BER performance comparison of decoding algorithms in a parallel TTCM scheme, AWGN channel, $K = 684$ symbols.

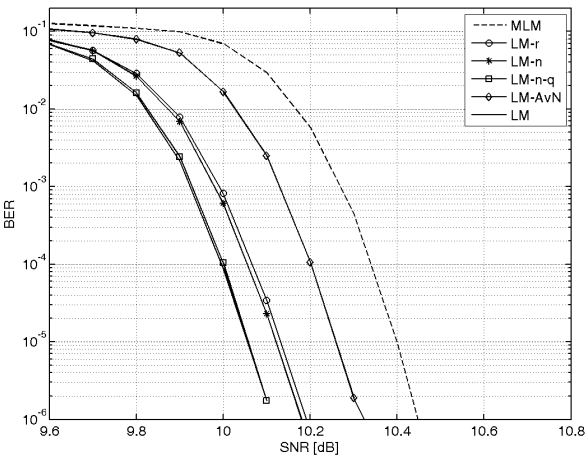

Fig. 3. BER performance comparison of decoding algorithms in a parallel TTCM scheme, AWGN channel, $K = 5000$ symbols.

The results for the Rayleigh fading channel are shown in Figs. 4 and 5. Here, the most of the simplified algorithms offer the Log-MAP performance except for the AvN Log-MAP and the Max-Log-MAP that are inferior by 0.2–0.3 dB at BER level of $10^{-3} - 10^{-4}$.

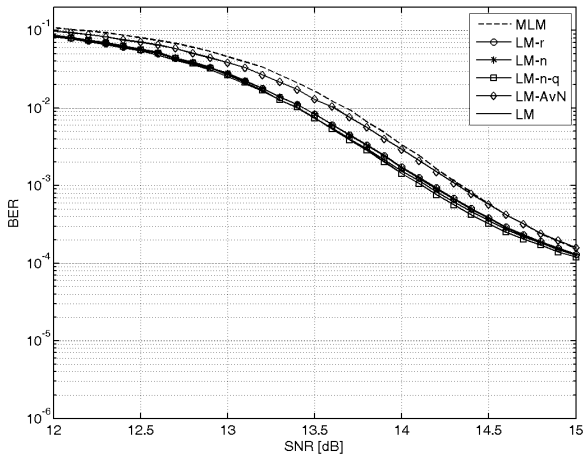

Fig. 4. BER performance comparison of decoding algorithms in a parallel TTCM scheme, uncorrelated Rayleigh fading channel, $K = 684$ symbols.

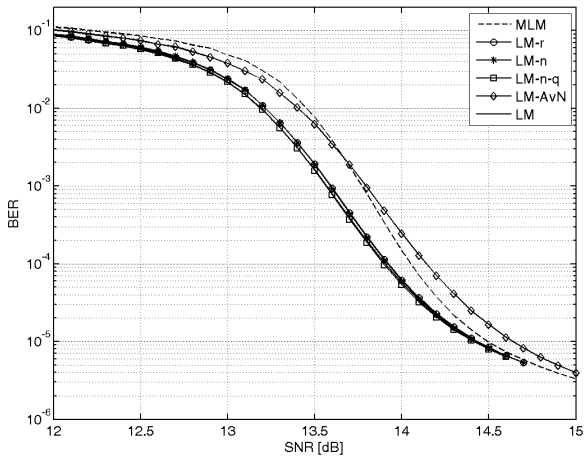

Fig. 5. BER performance comparison of decoding algorithms in a parallel TTCM scheme, uncorrelated Rayleigh fading channel, $K = 5000$ symbols.

Figs. 6 and 7 illustrate BER performance results for serial TTCM scheme in the AWGN channel and Figs. 8 and 9 in the Rayleigh fading channel. For the AWGN channel, it is observed that the LM-n-q algorithm is merely about 0.1 dB inferior to the Log-MAP and outperforms by 0.1–0.2 dB the LM-n and LM-r algorithms at BER of 10^{-5} for both block lengths. The AvN Log-MAP algorithm is less than 0.15 dB worse than the latter algorithms at the same BER level.

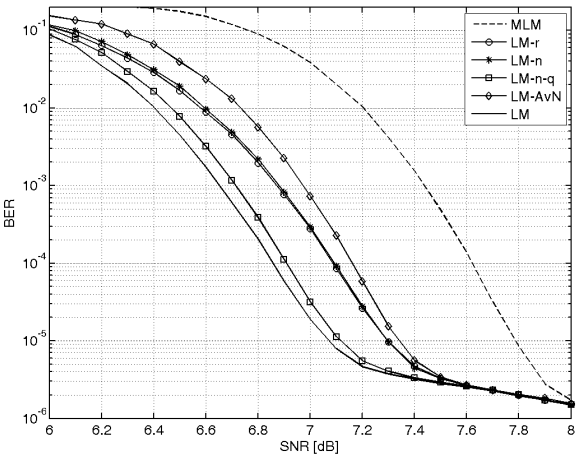

Fig. 6. BER performance comparison of decoding algorithms in a serial TTCM scheme, AWGN channel, $K = 684$ symbols.

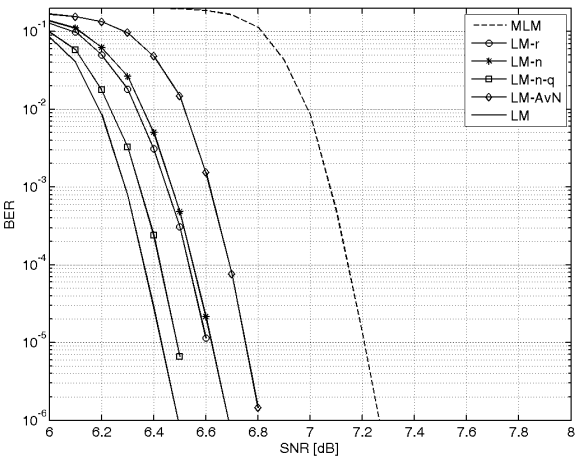

Fig. 7. BER performance comparison of decoding algorithms in a serial TTCM scheme, AWGN channel, $K = 5000$ symbols.

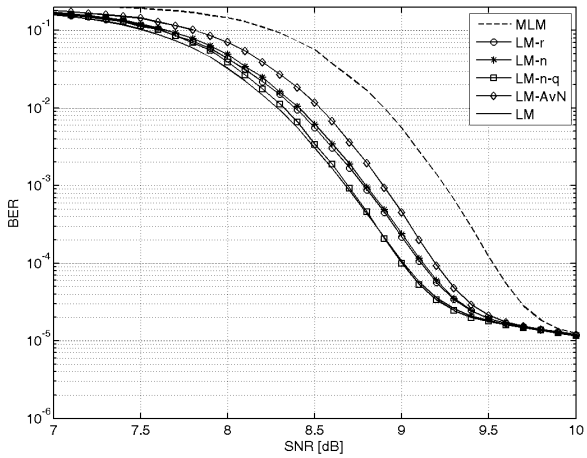

Fig. 8. BER performance comparison of decoding algorithms in a serial TTCM scheme, uncorrelated Rayleigh fading channel, $K = 684$ symbols.

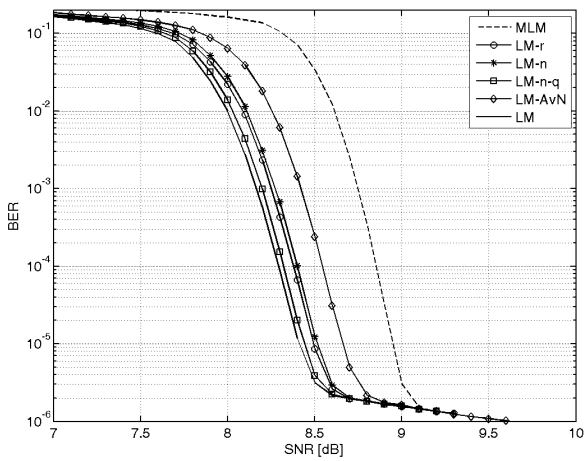

Fig. 9. BER performance comparison of decoding algorithms in a serial TTCM scheme, uncorrelated Rayleigh fading channel, $K = 5000$ symbols.

Comparing to the Max-Log-MAP algorithm, the performance of the AvN Log-MAP is improved by almost 0.5 dB. In the case of transmission over a Rayleigh fading channel, it can be noted that the LM-n-q algorithm achieves almost the Log-MAP performance in the whole range of simulated SNRs. The LM-n and LM-r algorithms reveal a loss of about 0.1 dB and the AvN Log-MAP of 0.2–0.3 dB to the Log-MAP at BER of $10^{-4} - 10^{-5}$. As for the AWGN channel, the AvN Log-MAP algorithm distinctly outperforms the Max-Log-MAP by 0.3 dB at the same BER level.

VI. Conclusions

Simplified decoding algorithms for SISO decoders in iterative receivers have gained considerable interest in recent years. Several algorithmic approaches have been proposed aiming for a simplification of the $\max^*$ operator and thus reducing the implementation complexity of the SISO decoders without a substantial loss of decoding performance. The use of algorithms with reduced computational requirements in iterative receivers is also favorable in terms of the energy efficiency which is an important issue in wireless tranceivers design. In this paper, some of these algorithms have been reviewed and compared in terms of computational requirements and error performance for the instance of turbo TCM transmission. In particular, complexity comparisons to the optimal Log-MAP algorithm reveal that significant reductions in the number of operations required are offered by the AvN Log-MAP, the LM-n, and the LM-n-q algorithms. For the TTCM schemes considered, a simplification of up to 41.1% per single decoding step has been obtained with the AvN Log-MAP algorithm. On the other hand, simulation results for AWGN and uncorrelated Rayleigh fading channels show that the performance degradation of these algorithms with respect to the Log-MAP is rather small and amounts to 0.0–0.3 dB, depending on the algorithm, TTCM scheme and the channel. The best BER performance among simplified algorithms examined achieves the LM-n-q algorithm. It should also be noted that similar results in terms of complexity and BER performance of the algorithms

have been obtained for other code rates and modulation sizes. Taking into account both computational requirements and error performance, one can conclude that the LM-n-q, the LM-n and the AvN Log-MAP algorithms are promising proposals for application in SISO decoders.

ACKNOWLEDGMENT

This work has been supported by the Polish Ministry of Science and Higher Education funds for the statute activity projects DS and DS-MK.

REFERENCES

[1] C. Berrou, A. Glavieux, P. Thitimajshima, "Near Shannon limit error-correcting coding and decoding: Turbo-codes", *Proc. IEEE Int. Conf. Commun. (ICC)*, Geneve, Switzerland, May 1993, 1064-1070.

[2] C. Berrou, A. Glavieux, "Near optimum error correcting coding and decoding: turbo-codes", *IEEE Trans. Commun.*, vol. 44, no. 10, 1996, 1261-1271.

[3] S. Benedetto, G. Montorsi, "Iterative decoding of serially concatenated convolutional codes", *Electron. Lett.*, vol. 32, no. 13, 1996, 1186-1188.

[4] R. G. Gallager, "Low-density parity-check codes", *IRE Trans. Inform. Theory*, vol. 8, no. 1, 1962, 21-28.

[5] G. Caire, G. Taricco, E. Biglieri, "Bit-interleaved coded modulation", *IEEE Trans. Inform. Theory*, vol. 44, no. 3, 1998, 927-946.

[6] P. Robertson, T. Wörz, "Bandwidth-efficient turbo trellis-coded modulation using punctured component codes", *IEEE J. Select. Areas Commun.*, vol. 16, no. 2, 1998, 206-218.

[7] A. Stefanov, T. M. Duman, "Turbo coded modulation for wireless communications with antenna diversity", *Proc. IEEE Vehicular Tech. Conf. (VTC)*, 1999, 1565-1569.

[8] H. Wymeersch, *Iterative Receiver Design*, Cambridge University Press, 2007.

[9] L. R. Bahl, J. Cocke, F. Jelinek, J. Raviv, "Optimal decoding of linear codes for minimizing symbol error rate", *IEEE Trans. Inform. Theory*, vol. IT-20, no.2, 1974, 284-287.

[10] P. Robertson, P. Höher, E. Villebrun, "Optimal and sub-optimal maximum a posteriori algorithms suitable for turbo decoding", *Eur. Trans. Telecomm.*, vol. 8, no. 2, 1997, 119-125.

[11] A. J. Viterbi, "An intuitive justification and a simplified implementation of the MAP decoder for convolutional codes", *IEEE J. Sel. Areas Commun.*, vol. 16, no. 2, 1998, 260-264.

[12] M. Sybis, P. Tyczka, S. Papaharalabos, P. T. Mathiopoulos, "Reduced-complexity algorithms for near-optimal decoding of turbo TCM codes", *IET Electron. Lett.*, vol. 45, no. 5, 2009, 278-279.

[13] M. Sybis, P. Tyczka, S. Papaharalabos, P. T. Mathiopoulos, G. Masera, M. Martina, "Log-MAP decoding of turbo codes and turbo trellis-coded modulation using piecewise-linear approximations of the max* operator", *Electronics and Telecommunications Quarterly*, vol. 55, no. 3, 2009, 435-451.

[14] M. Sybis, "Log-MAP equivalent Chebyshev inequality based algorithm for turbo TCM decoding", *IET Electron. Lett.*, vol. 47, no. 18, 2011, 1049-1050.

[15] S. Papaharalabos, M. Sybis, P. Tyczka, P. T. Mathiopoulos, "Novel generalized max* approximation method for simplified decoding of turbo and TTCM codes", *Wireless Pers. Commun.*, vol. 69, iss. 1, 2013, 373-386.

[16] M. Sybis, P. Tyczka, "Reduced complexity Log-MAP algorithm with Jensen inequality based non-recursive max* operator for turbo TCM decoding", *EURASIP Journal on Wireless Communications and Networking*, 2013:238.

[17] K.-L. Hsiung, S.-J. Kim, S. Boyd, "Tractable approximate robust geometric programming", *Spinger Optim. Eng. Journal*, vol. 9, no.2, 2008, 95-118.

[18] C. B. Schlegel, L. C. Perez, *Trellis and Turbo Coding*, Wiley and Sons, Inc., 2004.