Hybrid dynamical system perspective of asynchronous cellular automaton neuron model

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Abstract—The asynchronous cellular automaton neuron model can realize a wide variety of neuron-like nonlinear behaviors. In this paper, hybrid dynamical system perspectives of the asynchronous cellular automaton neuron model are discussed. Also, a multi-compartment soma-dendrite model based on the asynchronous cellular automaton neuron model is introduced.

1. Introduction

It is no exaggeration to say that the brain and the neuron are ones of the most sophisticated nonlinear dynamical systems. Many hardware models of neurons have been presented so far and their clinical and engineering applications have been also investigated intensively (see [1] and references therein). Major hardware neuron modeling approaches include the following ones (see also Table 1).

- An *analog nonlinear circuit* approach that uses a *nonlinear ordinary differential equation* (ab. ODE) to model the nonlinear dynamics of a neuron.
- A *switched capacitor* approach that uses a *nonlinear difference equation* to model the nonlinear dynamics of a neuron.
- A *digital processor* approach that uses a *numerical integration* to model the nonlinear dynamics of a neuron.
- A *synchronous sequential logic* approach that uses a traditional synchronous *cellular automaton* to model the nonlinear dynamics of a neuron [1]-[11].
- An *asynchronous sequential logic* approach that uses an *asynchronous cellular automaton* to model the nonlinear dynamics of a neuron.

Advantages and significances of the asynchronous cellular automaton neuron model (e.g., low hardware cost and dynamic reconfigurable capability) are discussed in an accompanying paper [12]. In this paper, rather than such advantages, hybrid dynamical system perspectives of the asynchronous cellular automaton neuron model are discussed. Also, a multi-compartment soma-dendrite model based on the asynchronous cellular automaton neuron model is introduced.

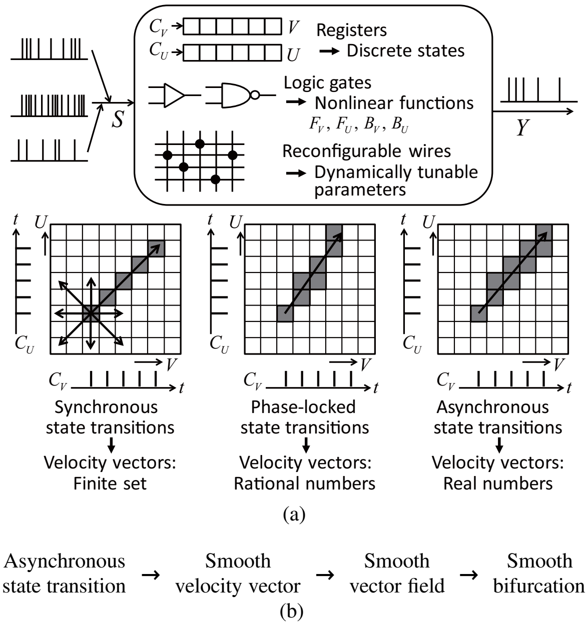


Figure 1: Concepts of the asynchronous cellular automaton neuron model. (a) The velocity vectors induced by synchronous state transitions are characterized by a set of finite integers. The velocity vectors induced by phase-locked state transitions are characterized by a set of finite rational numbers. The velocity vectors induced by asynchronous state transitions are characterized by a set of real numbers. (b) The asynchronous transitions of the discrete states realize a smooth velocity vector, a smooth vector field, and thus a smooth bifurcation.

2. Asynchronous cellular automaton neuron model

Fig. 1 illustrates concepts of the asynchronous sequential logic neuron model. Note that this model has two discrete states (V, U) but it can be generalized to a model with any number of discrete states. The two-state model in Fig. 1 consists of the following elements.

- Registers that are responsible for storing discrete states, e.g., a membrane potential $V \in \{0, 1, \cdots, N-1\} = \mathbf{Z}_N$ and a recovery variable $U \in \{0, 1, \cdots, M-1\} = \mathbf{Z}_M$.

Table 1: Hardware-oriented Neuron Modeling Approaches.
*The asynchronous cellular automaton has discrete states and continuous state transition times.

Hardware	Time and State	Dynamics	Control parameter
Analog nonlinear circuit	Continuous time Continuous state	Nonlinear ODE	Nonlinearity of circuit element such as MOSFET (not suited for on-chip learning)
Switched capacitor	Discrete time Continuous state	Iterative map	
Digital processor	Discrete time Discrete state	Numerical integration (hardware resource consuming)	Coefficient in digitally implemented nonlinear function
Synchronous sequential logic		Traditional cellular automaton	Wiring pattern among registers and logic gates (suited for on-chip learning)
Asynchronous sequential logic	Continuous time* Discrete state	Asynchronous cellular automaton	

Synonyms of "asynchronous cellular automaton " from some perspectives:

- Asynchronous sequential logic (hardware perspective)
- Asynchronous cellular automaton (dynamical system perspective)
- Asynchronous numerical integration (computation perspective)
- Asynchronous bifurcation processor (processor perspective)

- Logic gates that are responsible for realizing nonlinear functions, e.g., functions $F_V : \mathbf{Z}_N \times \mathbf{Z}_M \to \{-1, 0, 1\}$, $F_U : \mathbf{Z}_N \times \mathbf{Z}_M \to \{-1, 0, 1\}$, $B_V : \mathbf{Z}_N \times \mathbf{Z}_M \to \mathbf{Z}_N$, and $B_U : \mathbf{Z}_N \times \mathbf{Z}_M \to \mathbf{Z}_M$.
- Reconfigurable wires that are responsible for parameterizing the nonlinear functions.
- State-dependent clocks that are responsible for triggering transitions of the discrete states, e.g., clocks $C_V(t, V, U)$ and $C_U(t, V, U)$ the instantaneous frequencies of which depend on the discrete states (V, U).
- Like a biological neuron, the model accepts a spike-train stimulation input $S(t)$ from other neurons.

For simplicity, let us introduce the following notations.

"↑" denotes "*a positive edge of a clock.*"
":=" denotes "*an instantaneous transition of a discrete state.*"

Then, some of the asynchronous sequential logic neuron models are described by the following formulas.

Subthreshold dynamics:

$$\begin{aligned} V &:= V + F_V(V, U) \quad \text{if} \quad C_V(t, V, U) = \uparrow, \\ U &:= U + F_U(V, U) \quad \text{if} \quad C_U(t, V, U) = \uparrow. \end{aligned} \tag{1}$$

Stimulation via chemical synapse:

$$V := V + W \quad \text{if } S(t) = \uparrow, \tag{2}$$

where $W \in \{\cdots, -1, 0, 1, \cdots\}$ is a synaptic weight.

Firing:

$$\begin{aligned} &(V, U) := (B_V(V, U), B_U(V, U)) \\ &\text{if} \quad (V, U) \in \mathbf{L} \quad \text{and} \quad C_V(t, V, U) = \uparrow, \end{aligned} \tag{3}$$

where $\mathbf{L} \subset \mathbf{Z}_N \times \mathbf{Z}_M$ is a threshold set, which can be regarded as a *firing threshold* of a neuron model.

Output:

$$Y(t) = \begin{cases} 1 & \text{if} \quad (V, U) \in \mathbf{L} \quad \text{and} \quad C_V(t, V, U) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

In summary, the dynamics of the asynchronous cellular automaton neuron model is described by Eqs. (1)-(4).

Remarks on the concepts of the model:

- If $C_V = C_U$, the subthreshold dynamics in Eq. (1) has some analogies with one-step explicit numerical integration formulas such as the forward Euler method. Hence, the asynchronous cellular automaton neuron model can be regarded as a special kind of asynchronous numerical integration (see also Table 1). In addition, the asynchronous cellular automaton neuron model can be regarded as a special kind of asynchronous processor, which is designed to reproduce typical bifurcations of neurons (see also Table 1).
- The asynchronous cellular automaton neuron model is designed to have a much smaller resolution of the discrete state space than the digital processor neuron model. However, as illustrated in Fig. 1, the asynchronicity of the state transitions in Eqs. (1)–(3) can realize a smooth vector field. Conceptually speaking, in order to realize a smooth vector field and a smooth bifurcation structure, the asynchronous cellular automaton neuron model *wisely utilizes the continuousness of the time axis*, whereas the digital processor neuron model *straightforwardly utilizes a high resolution discrete state space*. This is the key design concept of the asynchronous cellular automaton neuron model.

3. Hybrid dynamical system perspective of asynchronous cellular automaton neuron model

For simplicity, let the clocks C_V and C_U be independent of the states (V, U):

$$C_V(t, V, U) = C_V(t), \quad C_U(t, V, U) = C_U(t).$$

Also, let the clocks $C_V(t)$ and $C_U(t)$ be periodic. Then, after the phase reduction [13], the dynamics of the phases φ_V and φ_U of the clocks $C_V(t)$ and $C_U(t)$ are described by the following equations.

Dynamics of phases of periodic clocks:

$$\frac{d\varphi_V}{dt} = \frac{2\pi}{T_V}, \quad \frac{d\varphi_U}{dt} = \frac{2\pi}{T_U},$$

where T_V and T_U are positive real numbers and are periods of the clocks $C_V(t)$ and $C_U(t)$, respectively. Then the periodic clock are described by the following equation.

Periodic clocks:

$$\begin{aligned} C_V(t) &= \uparrow \quad \text{if } \varphi_V(t) = 0 \pmod{2\pi}, \\ C_U(t) &= \uparrow \quad \text{if } \varphi_U(t) = 0 \pmod{2\pi}. \end{aligned} \tag{5}$$

Now the whole system (i.e., the asynchronous cellular automaton neuron model in Eqs. (1)-(4) and the clock generators in Eq. (5)) has the following states.

$$\begin{aligned} \text{Discrete states:} &\quad V \in \mathbf{Z}_N, \quad U \in \mathbf{Z}_M, \\ \text{Continuous states:} &\quad \varphi_V \in [0, 2\pi), \quad \varphi_U \in [0, 2\pi), \\ \text{Whole state space:} &\quad \mathbf{S} = \mathbf{Z}_N \times \mathbf{Z}_M \times [0, 2\pi) \times [0, 2\pi). \end{aligned}$$

For simplicity, let the firing threshold $\mathbf{L}$ be

$$\mathbf{L} = \{(V, U) | V = N - 1\}.$$

Then, referring to Eq. (3), the asynchronous cellular automaton neuron model fires when

$$V = N - 1, \quad \varphi_V = 0 \pmod{2\pi}.$$

Assuming the model continues to fire, the following Poincare section $\mathbf{\Sigma}$ can be defined[1].

$$\mathbf{\Sigma} = \{(V, U, \varphi_V, \varphi_U) | V = N - 1, \ \varphi_V = 0 \pmod{2\pi}\} \subset \mathbf{S}.$$

Let the states U and φ_U at the n-th firing moment (i.e., the n-th moment when the state vector $(V, U, \varphi_V, \varphi_U)$ visits the Poincare section $\mathbf{\Sigma}$) be denoted by u_n and φ_n, respectively. Then the dynamics of the states (u_n, φ_n) is described by the following discrete-continuous hybrid return map.

$$(u_{n+1}, \varphi_{n+1}) = F(u_n, \varphi_n), \quad F : \mathbf{\Sigma} \to \mathbf{\Sigma}.$$

[1]If the Poincare section $\mathbf{\Sigma}$ is appropriately defined in another way, this assumption is not needed. In this case, not only spiking behaviors but also subthreshold behaviors (e.g., resting state and subthreshold oscillation) of the model can be analyzed by the hybrid return map in Eq. (6) [1].

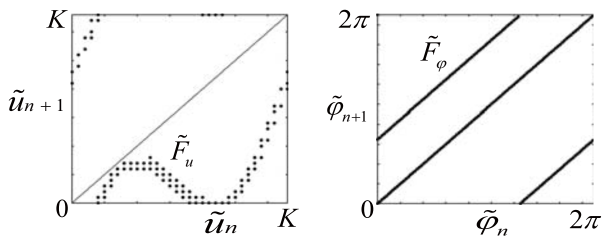


Figure 2: An example of the discrete-continuous hybrid iterative map $(\tilde{F}_u, \tilde{F}_\varphi)$ [5], which is equivalent to the map (F_u, F_φ) in Eq. (6).

For better understanding, rather than the above formula, let us use the following formula.

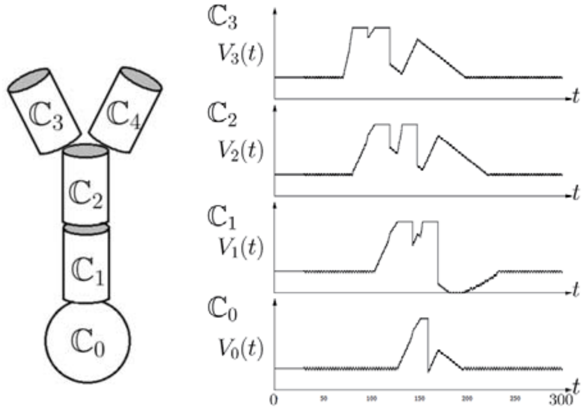
Discrete-continuous hybrid return map:

$$\begin{aligned} u_{n+1} &= F_U(u_n, \varphi_n), \quad F_u : \mathbf{\Sigma} \to \mathbf{Z}_M, \\ \varphi_{n+1} &= F_\varphi(u_n, \varphi_n), \quad F_\varphi : \mathbf{\Sigma} \to [0, 2\pi). \end{aligned} \tag{6}$$

Fig. 2 shows a discrete-continuous hybrid return map, which is equivalent to that in Eq. (6).

Remarks on significances of the asynchronicity:

- If $T_U/T_V = 1$, the clocks $C_V(t)$ and $C_U(t)$ exhibit a 1:1 synchronization. This corresponds to the situation in Fig. 1(a). In this case, the orbit $(\varphi_1, \varphi_2, \cdots)$ of the continuous state variable φ_n is restricted in a point or a set of few points.
- If T_U/T_V is a rational number, the clocks $C_V(t)$ and $C_U(t)$ exhibit an m:n synchronization. This corresponds to the situation in Fig. 1(b). In this case, the orbit $(\varphi_1, \varphi_2, \cdots)$ of the continuous state variable φ_n is restricted in a set of many points.
- If T_U/T_V is a irrational number, the clocks $C_V(t)$ and $C_U(t)$ exhibit a quasi-periodic behavior. This corresponds to the situation in Fig. 1(c). In this case, the orbit $(\varphi_1, \varphi_2, \cdots)$ of the continuous state variable φ_n lies densely in a continuous subset of the set $[0, 2\pi)$.
- The measure of the parameter set for the case where T_U/T_V is rational or integer is zero. Hence, real electronic circuits of the clock generators should have an irrational T_U/T_V.
- The discrete state variable u_n of the hybrid return map F determines the position of the discrete state vector (V, U) in its phase space $\mathbf{Z}_N \times \mathbf{Z}_M$ and thus the discrete state variable u_n determines the dominant behavior of the asynchronous cellular automaton neuron model. On the other hand, the continuous state variable φ_n of the hybrid return map F exhibits a quasi-periodic behavior for the case of the asynchronous clocks (i.e., for the case of the irrational T_U/T_V) and then it acts as a perturbation to the discrete state variable u_n. This perturbation realizes the smoothness of the map F_u



(a) (b)

Figure 3: (a) Multi-compartment soma-dendrite model based on the asynchronous cellular automaton neuron model [10]. (b) Forward propagation of action potentials from the dendritic compartment C_3 to the soma compartment C_0.

as shown in Fig.2 and thus realizes the smooth vector field and the smooth bifurcation structure as illustrated in Fig. 1.

4. Multi-Compartment Soma-Dendrite Model

Fig. 3(a) shows a multi-compartment soma-dendrite model based on the asynchronous cellular automaton neuron model, where C_0 is a soma compartment and $\{C_1, C_2, C_3, C_4\}$ are dendritic compartments [10]. Fig. 3(b) shows a forward propagation of action potentials from the dendritic compartment C_3 to the soma compartment C_0. Due to smoothness of the vector field of the compartments $\{C_0, C_1, C_2, C_3, C_4\}$ realized by asynchronous clocks, the hardware cost (i.e., the number of configuration block occupied in a Xilinx FPGA) of the multi-compartment soma-dendrite model based on the asynchronous cellular automaton neuron model becomes almost 1/7 compared to a multi-compartment soma-dendrite model based on the Izhikevich simple neuron model [14] (see [10] for detailed comparisons of the hardware cost).

5. Conclusions

In this paper, the hybrid dynamical system perspectives of the asynchronous cellular automaton neuron model were discussed. Also, the multi-compartment soma-dendrite model based on the asynchronous cellular automaton neuron model was introduced. It was shown that, due to the asynchronicity of the clocks, the multi-compartment soma-dendrite model can have the smooth vector field and consumes less hardware resources compared to the ODE neuron model. This work was supported by KAKENHI Grant Number 20318603.

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