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[Invited Talk]

Cluster Synchronization, Desynchronization, and Symmetries in Complex Networks of Oscillators

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Abstract–Exact synchronization of oscillators is important in many types of networks, e.g. power distribution, telecommunication, neuronal and biological networks. Many networks are observed to produce patterns of synchronized clusters, but it has been difficult to predict these clusters or understand the conditions under which they form. Here we develop techniques for using mathematical group theory for the analysis of network dynamics that shows the connection between network symmetries and cluster formation. We confirm our results experimentally. We observe a surprising phenomenon in which some clusters lose synchrony without disturbing the others. Our analysis is general in that such behavior will occur in a wide variety of networks and node dynamics. The results could lead to new understanding of the dynamical behavior of networks ranging from neural to social.

1. Introduction

Global complete synchronization of all identical oscillators (even chaotic ones) in networks has been studied for many years and the development of the master stability function [1] has facilitated the understanding of the stability of the fully synchronized state even in very complex networks. Recently studies of cluster synchronization (CS) patterns of networks of oscillators has shown very interesting situations where global synchronization does not exist, but groups (clusters) of synchronized nodes do – although nodes in one cluster are not synchronized with those in another [2-7] . However these studies have often focused on networks with special topology or arrangements of oscillators with specifically set time delays in the coupling to induce cluster patterns of synchronization.

We analyze the more common case where the intrinsic network symmetries are neither intentionally produced nor easily discerned. We present a comprehensive treatment of CS, which uses the tools of computational group theory to reveal the hidden symmetries of networks and predict the patterns of synchronization that can arise. We use irreducible group representations to find a block diagonalization of the variational equations that can predict the stability of the clusters. We further observe and explain a generic symmetry-breaking bifurcation termed isolated desynchronization, in which one or more clusters lose synchrony while the remaining clusters stay synchronized. The analytical results are confirmed through experimental measurements in a spatiotemporal electro-optic network. Throughout the text, we use the abbreviation ID for isolated desynchronization.

2. Analysis of Cluster Formation from Symmetries

2.1. Dynamical Equations of Motion

We assume the following form of dynamics in a network,

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i(t)) + \sigma \sum_{j=1}^{N} C_{ij}\mathbf{H}(\mathbf{x}_j), \quad i = 1,\ldots,N \qquad (1)$$

where $\mathbf{x}_i$ is the n-dimensional state vector of the ith oscillator, $\mathbf{F}$ describes the dynamics of each oscillator, C is a coupling matrix that describes the connectivity of the network, σ is the overall coupling strength and $\mathbf{H}$ is the output function of each oscillator. Equation (1) or its equivalent forms provide the dynamics for many networks of oscillators, including all those in refs [2-8]. The form of Eq.(1) for an iterated system can also be analyzed with the methods we show here. More general forms of Eq.(1) are possible [9], but the analysis is essentially the same as here. Our coupling matrix is chosen to be the network's adjacency matrix A plus a scalar multiple (β) of the $N\times N$ identity matrix $\mathbf{1}_N$. Other diagonal terms can also be treated with our methods. For example, our approach will carry through with Laplacian coupling, but we want to avoid global synchronization here to focus on the clusters and we think most networks, especially natural ones will not be tuned to have self-coupling that perfectly cancels input from other oscillators in the network like a Laplacian weighting.

2.2. Finding Symmetries in the Network Dynamics

The symmetries of a network (also called the automorphisms [10,11] of the adjacency matrix) are those permutations of nodes and associated edges which leave the network (adjacency matrix) unchanged (invariant). The symmetries of the network form a (mathematical) group G. Each symmetry g of the group can be described by a permutation matrix R_g that leaves the dynamical equations unchanged (that is, each R_g commutes with A

and thus C). The set of symmetries a network can be quite large, even for small networks, but it can be calculated from C using widely available discrete algebra routines [12,13]. Figure 1a shows three graphs generated by randomly removing 5 or 7 edges from an otherwise fully connected 9-node network. Although the graphs appear similar and exhibit no obvious symmetries, the first instance has no symmetries (other than the identity permutation), while the others have 16 and 480 symmetries, respectively. Thus, for even a moderate number of nodes, finding the symmetries can become impossible by inspection.

2.3. Identifying Clusters of Synchronization

Because the equations of motion (Eq.(1)) are invariant under the symmetry operations oscillators (nodes) related by symmetries will remain synchronized with each other if started in a synchronized state within the cluster. Dynamically speaking this means the cluster synchronized states are flow invariant in Eq.(1). To find the clusters we only need find which nodes are mapped into each other. This is given by the orbits of the group [10] which are provided by the computer group computations [12,13]. If there are M clusters, then there are M synchronized states of motion $\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}$. All oscillators in the jth cluster have motion $\mathbf{s}_j(t)$. Let K_m be the set of nodes (oscillators) in the mth cluster.

2.4. Stability of the Cluster States

The stability of the cluster states is determined by the solutions to the variational equations for Eq.(1). These essentially give the dynamics of small perturbations to the synchronized states and are derived from Eq.(1). These are obtained as,

$$\delta\dot{\mathbf{x}}_i = \left\{ \sum_{m=1}^{M} E^{(m)} \otimes \left[D\mathbf{F}(\mathbf{s}_m(t)) + \sigma C_{ij} D\mathbf{H}(\mathbf{s}_m(t)) \right] \right\} \cdot \delta\mathbf{x}_i \tag{2}$$

where the Nn-dimensional vector $\delta\mathbf{x} = \left[\delta\mathbf{x}_1^T, \delta\mathbf{x}_2^T, \ldots, \delta\mathbf{x}_N^T\right]$ and $E^{(m)}$ is an N-dimensional diagonal matrix with values,

$$E_{ii}^{(m)} = \begin{cases} 1, \text{ if } i \in K_m \\ 0, \text{ otherwise} \end{cases} \tag{3}$$

We cannot use Eq.(2) as it stands to find the stability of the clusters. This is because the perturbations (components of $\delta\mathbf{x}$) which are parallel to the synchronized states are mixed into perturbations which are transverse to the synchronization manifold [14] as well as we do not have each cluster separated from the others. In order to unmix the perturbations we have to perform a transformation T on the coupling matrix C which block diagonalizes it and does the separation of perturbations. In Ref. [14]. we outline how to set up this transformation. We end up with a block diagonalized matrix B=TCT^{-1} which decomposes variational equations to,

$$\dot{\boldsymbol{\eta}} = \left[\sum_{m=1}^{M} J^{(m)} \otimes D\mathbf{F}(\mathbf{s}_m(t)) + \sigma B \otimes I_n \sum_{m=1}^{M} J^{(m)} \otimes D\mathbf{H}(\mathbf{s}_m(t)) \right] \tag{4}$$

where $\boldsymbol{\eta} = T \otimes I_n \delta\mathbf{x}$, $J^{(m)} = TE^{(m)}T^{-1}$, and B is the block diagonalization of C. For more details see Ref. [14]. Fig.1 (c) shows B for the three cases. Note that the upper-left block is associated with the synchronized state and determines the dynamics of the perturbations parallel to that state, i.e. they do not affect the synchronization. Hence, we ignore this block. The other blocks are transverse to the synchronized state and

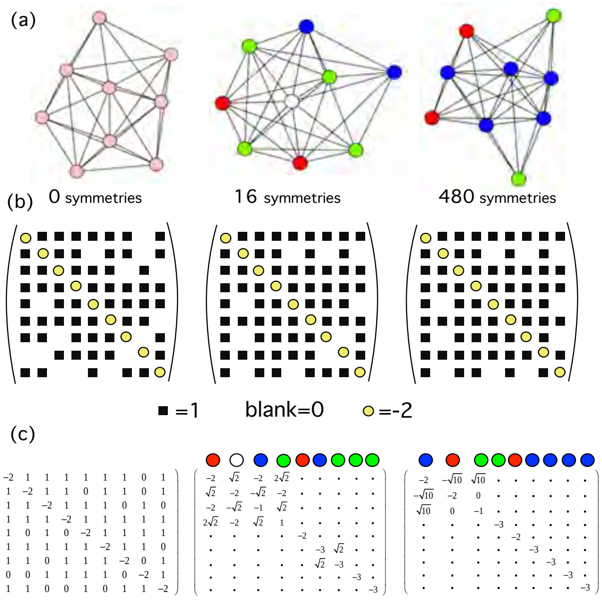

Fig.1 Three realizations of networks generated by starting with a fully (all-to-all) connected network and randomly removing 7 edges (for the 0 symmetry case) and 5 edges for the 16 and 480 symmetry cases. Part (a) shows the clusters in the networks discovered by using symmetry analysis software [12,13]. Part (b) shows the original coupling matrices (C) where the diagonal values are β= –2. Part (c) shows the block-diagonalized coupling matrices which are used to determine the stability of the cluster states shown. Note that the 16 symmetry system has a transverse 2 ×2 block indicating an intertwined pair of clusters. The colors above the columns show which clusters are associated with each row and column.

determine the stability of the cluster states. The motion we will consider in our experiment is chaotic so we use Eq.(4) to calculate Lyapunov exponents for our experimental system. For fixed point motion one would use eigenvalues and for periodic motion Floquet exponents or multipliers would be appropriate.

3. Electro-optic Experiments

Figure 2a shows the optical system used to study cluster synchronization. Light from a 1,550-nm light-emitting diode passes through a polarizing beamsplitter and quarter wave plate, so that it is circularly polarized when it reaches the spatial light modulator (SLM). The SLM surface imparts a programmable spatially dependent phase shift x between the polarization components of the reflected signal, which is then imaged, through the polarizer, onto an infrared camera [15]. The relationship between the phase shift x applied by the SLM and the normalized intensity **I** recorded by the camera is $\mathbf{I}(x)=(1-\cos x)/2$. The resulting image is then fed back through a computer to control the SLM. More experimental details are given in the Methods section.

The dynamical oscillators that form the network are realized as square patches of pixels selected from a 32×32 tiling of the SLM array. Figure 2b shows an experimentally measured camera frame captured for an 11-node random network generated by randomly deleting 9 edges from the fully connected network (see the inset of Fig. 3). The phase shift of the ith region, x_i, is updated iteratively according to:

$$x_i^{t+1} = \left[\beta\mathbf{I}(x_i^t) + \sigma\sum_j C_{ij}\mathbf{I}(x_j^t) + \delta\right] \bmod 2\pi \qquad (5)$$

where β is the self feedback strength, and the offset δ is introduced to suppress the trivia solution $x_i = 0$. Eq. (5) is a discrete time equivalent of Eq. (1). Depending on the values of β, σ and δ, Eq. (5) can show constant, periodic or chaotic dynamics. There are no experimentally-imposed constraints on the adjacency matrix C_{ij} , which makes this system an ideal platform to explore synchronization in complex networks.

Fig, 3 plots the time-averaged root-mean square (RMS) synchronization error for all four of the nontrivial clusters shown in Fig. 2b, as a function of the feedback strength , holding σ constant. We find qualitatively similar results if σ is chosen as a bifurcation parameter with held constant.

For the case of $\beta = 0.72\pi$ the system clearly partitions into 4 stable synchronized clusters plus one unsynchronized node. At $\beta = 1.4\ \pi$, the magenta colored cluster (see Fig.3), which contains four nodes, has split into two smaller clusters of 2 nodes each, while the other two clusters remain synchronized. This illustrates two examples of a bifurcation commonly seen in our experiments and simulations isolated desynchronization, where one or more clusters lose stability, while all others remain synchronized. At $\beta = 1.76$, two clusters, shown in Fig. 1 as red and blue, undergo isolated desynchronization together. In Fig. 3a, the synchronization error curves for these two clusters are visually indistinguishable. The synchronization of these two clusters is intertwined: they will always either synchronize together or not at all. Each of the two nodes in the blue cluster is coupled to exactly one node in the red cluster. If the blue cluster is not synchronized, the red cluster cannot synchronize because its two nodes are receiving different input. The group analysis treats this automatically and yields a transverse 2×2 similar to the case in Fig. 1 (c).

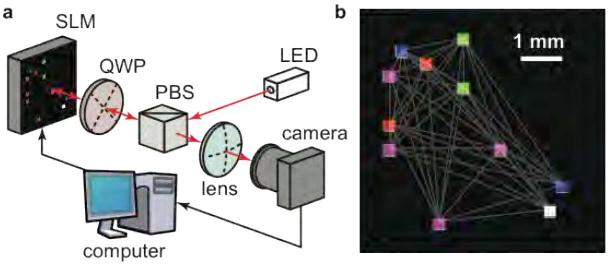

FIG. 2: Experimental conguration. a) Light is reected from the SLM, and passes though polarization optics, so that the intensity of light falling on the camera is modulated according the phase shift introduced by the SLM. Coupling and feedback are implemented by a computer. b) An image of the SLM recorded by the camera in this configuration. Oscillators are shaded to show which cluster they belong to, and the connectivity of the network is indicated by superimposed gray lines. The phase shifts applied by the square regions are updated according to equation (5).

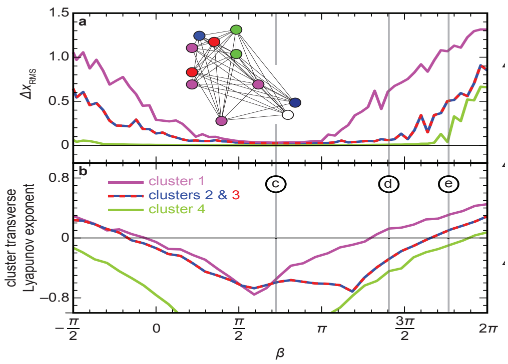

Fig. 3. a) Cluster synchronization error as the self feedback, is varied. For all cases considered, δ = 0:525 and σ = 0:6π. Inset shows the network with clusters colored to match the plots. b) Maximum Lyapunov exponent calculated from simulation agrees with synchronization pattern in the experiment.

The isolated desynchronization bifurcations we observe are predicted by computation of the maximum Lyapunov exponent (MLE) of the transverse blocks of Eq. (4), shown in Fig. 3. The region of stability of each cluster is predicted by a negative MLE. There are only three MLEs: the two intertwined clusters are described by a 2-dimensional block in the block-diagonalized coupling matrix B. These stability calculations reveal the same bifurcations as seen in the experiment.

In Ref. [14] we show that the phenomenon of isolated desynchronization can be explained by using a group decomposition of the network symmetry group into a direct product of subgroups [16,17] each of which act solely on one or more clusters, but not on others. This leads to the situation where one cluster can desynchronize, but other clusters receiving input from the desynchronized cluster receive only the sum of outputs from that cluster. Hence, symmetries of the subgroup only rearrange the sum terms and each node (oscillator) of the former cluster still gets the same sum and their motion has flow invariance to a synchronized initial condition.

In Ref. [14] we show that the techniques in this paper can also be applied to electric power grids so we can identify potential clusters that can form in those grids if the global synchronization of generators is lost.

The phenomena of symmetry-induced cluster synchronization and ID appear to be possible in many model, man-made, and natural networks, at least when modeled as unweighted couplings and identical systems. This is because our work here and that of Refs. [16,17] show that many types of networks can have symmetries and Refs.[8,18] show that many man-made and natural networks have dynamics similar to or reducible to Eq.(1) or its generalizations mentioned above. We've shown that ID is explained generally as a manifestation of clusters and subgroup decompositions. Furthermore, computational group theory can greatly aid in identifying cluster synchronization in complex networks where symmetries are not obvious or far too numerous for visual identification. It also enables explanation of types of desynchronization patterns, and transformation of dynamic equations into more tractable forms. This leads to an encompassing of or overlap with other phenomena which are usually presented as separate. This list includes (1) remote synchronization [19] in which nodes not directly connected by edges can synchronize (this is just a version of cluster synchronization), (2) some types of chimera states [15,20] which can appear when the number of trivial clusters is large and the number of nontrivial clusters is small, but the clusters are big (see [21] for some simple examples), (3) partial synchronization where only part of the network is synchronized (shown for some special cases in[22]).

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