Estimation of delay time in time-delayed feedback systems

Vladimir I. Ponomarenko[†], Mikhail D. Prokhorov[†] and Vladimir S. Khorev[‡]

† Saratov Branch of Kotel'nikov Institute of Radio Engineering and Electronics of Russian Academy of Sciences, Zelyonaya Street, 38, Saratov 410019, Russia
‡ Department of Nano- and Biomedical Technologies, Saratov State University, Astrakhanskaya Street, 83, Saratov 410012, Russia
Email: ponomarenkovi@gmail.com, mdprokhorov@yandex.ru, zergs@inbox.ru

Abstract– We propose a new method for the recovery of delay time from time series of time-delay systems based on the nearest neighbour analysis. The method allows one to reconstruct delays in time-delay systems described by the first- and second-order delay-differential equations. It can be applied to time series heavily corrupted by additive and dynamical noise.

1. Introduction

Self-sustained oscillators with delay-induced dynamics are highly widespread in nature. Their abundance results from such fundamental features as the finite velocity of signal propagation that is especially displayed in spatially extended systems [1] and time-delayed feedback inherent in many physical [2,3], chemical [4], climatic [5], and biological [6] systems and processes. Studying time-delay systems it is important to know the delay times whose values in many respects define the system dynamics and features. Knowledge of delay times is of considerable significance in model construction and prediction of system behaviour in time and under parameter variation. That is why the problem of delay time reconstruction from experimental time series attracts a lot of attention.

To solve this problem a variety of methods has been proposed, which allows one to recover the delay times of time-delayed feedback systems from their chaotic time series. Many of these methods are based on the projection of the infinite-dimensional phase space of time-delay systems onto low-dimensional subspaces [7–9]. They use different criteria of quality for the system reconstruction, for example, the minimal forecast error of the constructed model [7], minimal value of information entropy [8], or various measures of complexity of the projected time series [9]. The methods of delay time recovery are known based on employment of regression analysis [10], statistical analysis of time intervals between extrema in the time series [11], information-theory approaches [12], multiple shooting approach [13], optimization algorithm [14], and adaptive synchronization [15]. A separate group of methods for delay time estimation is based on the analysis of the time-delay system response to external perturbations [16]. These methods can be applied to systems performing not only chaotic, but also periodic oscillations.

In this paper we propose a novel method for recovering delay time from time series. It is based on the nearest neighbour method. The method of nearest neighbours is widely used in different scientific disciplines for nonlinear time series analysis [17]. Its main areas of application are classification of objects and forecast of time series. In the object classification problem the basic idea of the nearest neighbour method is that the object is assigned to the class of its nearest neighbour or to the class most common amongst its k nearest neighbours. In application to the forecast of a time series the method idea is to use for prediction of a future state of a system its states in the past, which are most similar to the current state. We propose using the nearest neighbour method for the first time for estimating the delay time of a delayed feedback system from time series.

The paper is organized as follows. In Section 2 we present the idea of the method and apply it to recover first-order time-delay systems with a single delay in chaotic and periodic regimes. In Sections 3 the method is applied for the reconstruction of delays in scalar time-delay systems of second order. In Section 4 we summarize our results.

2. Recovery of delay time in first-order time-delay systems

Let us explain the method idea with one of the most popular first-order delay-differential equation with a single delay:

$$\varepsilon \dot{x}(t) = -x(t) + f\left(x(t-\tau)\right), \qquad (1)$$

where τ is the delay time, the parameter ε characterizes the inertial properties of the system, and f is a nonlinear function. Note that the Mackey-Glass equation [6] and the Ikeda equation [1], which became standard equations in the study of time-delay systems, can be reduced to Eq. (1).

Analyzing time series, we always deal with variables measured at discrete instants of time. Therefore, it is convenient to pass from differential Eq. (1) to the difference equation

$$\varepsilon \frac{x(t+\Delta t) - x(t)}{\Delta t} = -x(t) + f\left(x(t-\tau)\right), \qquad (2)$$

where Δt is the sampling time. Equation (2) can be rewritten as

$$x(t+\Delta t) = a_1 x(t) + a_2 f\left(x(t-\tau)\right), \qquad (3)$$

where $a_1 = 1 - \Delta t/\varepsilon$ and $a_2 = \Delta t/\varepsilon$. Let us write Eq. (3) in the form of the discrete-time map

$$x_{n+1} = a_1 x_n + a_2 f(x_{n-d}), \qquad (4)$$

where $n = t/\Delta t$ is the discrete time and $d = \tau/\Delta t$ is the discrete delay time.

Assume that we have a time series $\{x_n\}_{n=1}^{N}$ from the system (1), where N is the number of points. Let us define vector $\vec{X}_i = (x_i, x_{i-d})$ and find vector $\vec{X}_j = (x_j, x_{j-d})$ with $j \neq i$, which is a nearest neighbour of $\vec{X}_i$. The nearest neighbour for a given vector can be chosen according to some metrics. The most widely used metrics is the Euclidean metrics

$$L\left(\vec{X}_i, \vec{X}_j\right) = \sqrt{\left(x_i - x_j\right)^2 + \left(x_{i-d} - x_{j-d}\right)^2}. \qquad (5)$$

The vector $\vec{X}_j$ will be the nearest neighbour of $\vec{X}_i$, if the distance $L\left(\vec{X}_i, \vec{X}_j\right)$ is minimal. Generally, it is a common practice to find not one, but k nearest neighbours for a given vector.

The basic idea of the proposed method is that the nearest neighbour vectors containing the system (4) dynamical variable at the instants of time n and $n-d$, where $n \in \left[d+1, N-1\right]$, will lead to the close states of the system at the instants of time $n+1$, because the system (4) evolution is defined by its current state and the state at the delayed instant of time. Since the delay time is a priori unknown, we vary the trial delay times m within some interval and for k nearest neighbours of each vector $\vec{X}_n = (x_n, x_{n-m})$ constructed from the time series estimate the variance σ_n^2 of the system states at the corresponding instants of time $n+1$.

In the case of false choice of m ($m \neq d$), the variance of these states may be great, because the system states at the instants of time $n+1$ do not depend on the system states at the instants of time $n-m$. True delay time d can be estimated as the value at which the minimum of the following dependence:

$$D(m) = \frac{1}{N-m-2} \sum_{n=m+1}^{N-1} \sigma_n^2 \qquad (6)$$

is observed.

We apply the method to time series of the Mackey-Glass equation

$$\dot{x}(t) = -bx(t) + \frac{ax(t-\tau)}{1 + x^c(t-\tau)}, \qquad (7)$$

which can be converted to Eq. (1) by division by b. The parameters of Eq. (7) are chosen to be $a = 0.2$, $b = 0.1$, $c = 10$, and $\tau = 300$ to produce a dynamics on a chaotic attractor. The sampling time is $\Delta t = 1$ and the number of points is $N = 10000$. Part of the time series is shown in Fig. 1(a).

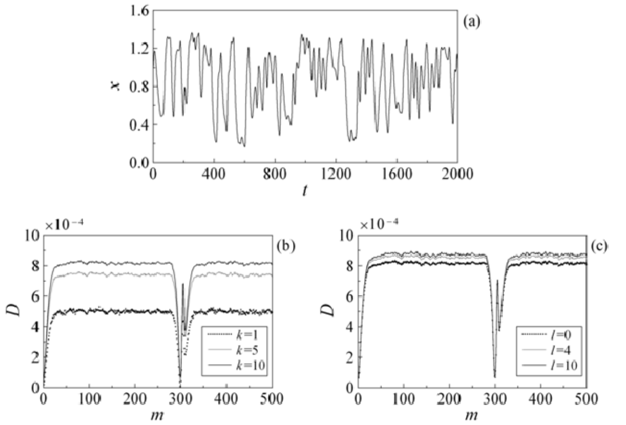

Fig. 1. (a) The time series of the Mackey-Glass equation in the chaotic regime. (b) Dependences of D on the trial delay time m for different numbers k of nearest neighbours. (c) Dependences $D(m)$ for different numbers l of close in time vectors excluded from consideration.

Figure 1(b) depicts the dependence of D on the trial delay time m for different numbers k of nearest neighbours for vector $\vec{X}_n = (x_n, x_{n-m})$. The value of m is varied from 1 to 500 with a step of 1. All the dependences $D(m)$ exhibit a well-pronounced absolute minimum at $m = 300$, which provides an accurate recovery of the discrete delay time $d = \tau/\Delta t = 300$.

If the time series points are sampled with a high frequency, a situation is possible in which the vectors $\vec{X}_j = (x_j, x_{j-d})$ with $j = i \pm p$ ($p = 1, 2, \ldots P$) that are close in time to vector $\vec{X}_i = (x_i, x_{i-d})$ will be detected as its nearest neighbours. To avoid this undesirable situation in the search for the nearest neighbours of vector $\vec{X}_i = (x_i, x_{i-d})$, one should exclude from consideration $l = 2P$ vectors $\vec{X}_j = (x_j, x_{j-d})$ close to $\vec{X}_i$ in time.

The dependences $D(m)$ are plotted in Fig. 1(c) for $k = 10$ and different numbers l of close in time vectors, which are not taken into account in searching for nearest neighbours. All the plots exhibit a sharp absolute minimum at $m = d = 300$, as well as the plots in Fig. 1(b).

It should be noted that instead of searching for a fixed number k of nearest neighbours for vector $\vec{X}_i = (x_i, x_{i-d})$, one can assign all vectors $\vec{X}_j = (x_j, x_{j-d})$ to its nearest neighbours, if $L\left(\vec{X}_i, \vec{X}_j\right) < \delta$, where δ is a small quantity. The plots of $D(m)$ constructed in this way of finding nearest neighbour vectors are similar to the plots presented in Fig. 1(b). The appropriate choice of the parameters k and δ enables one to achieve almost complete coincidence of the results of searching for nearest neighbours in both ways. In addition, we have found that the choice of the metrics for searching nearest neighbours has almost no effect on the form of the dependences *D*(*m*).

To test the method efficiency in the presence of noise we apply it to the data produced by adding a zero-mean Gaussian white noise to the time series of Eq. (7). The obtained results are presented in Fig. 2(a) for different levels of additive noise at $k = 10$ and $l = 10$. The location of the minimum of *D*(*m*) allows us to recover the delay time accurately even for noise level of about 30% (the signal-to-noise ratio is about 10 dB). Such level of noise greatly exceeds the noise level that is allowed for applying most of other methods of delay time reconstruction.

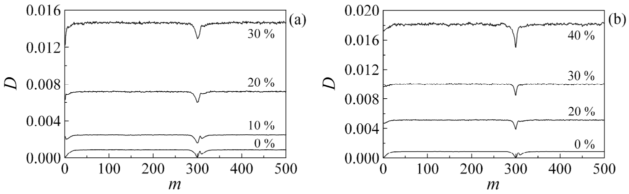

Fig. 2. Dependences *D*(*m*) for the Mackey-Glass system in the chaotic regime for different levels of additive noise (a) and dynamical noise (b). The levels of noise are indicated in % near the corresponding curves.

The proposed method is even more robust with respect to the dynamical noise. In Fig. 2(b) the dependences *D*(*m*) are shown at $k = 10$ and $l = 10$ for the case, where a zero-mean Gaussian white noise is added to the right-hand side of Eq. (7). In all the plots constructed in Fig. 2(b) for different levels of noise the minimum of *D*(*m*) is observed at $m = 300$.

3. Recovery of delay time in second-order time-delay systems

The proposed method can be easily extended to high-order time-delay systems. In particular, it can be modified for the systems described by the second-order delay-differential equations

$$\varepsilon_2 \ddot{x}(t) + \varepsilon_1 \dot{x}(t) = F\left(x(t), x(t-\tau)\right), \quad (8)$$

where ε_1 and ε_2 are the parameters characterizing the inertial properties of the system. As an example we consider the following system:

$$\varepsilon_2 \ddot{x}(t) + \varepsilon_1 \dot{x}(t) = -x(t) + f\left(x(t-\tau)\right). \quad (9)$$

Using the described above formalism, one can pass from differential Eq. (11) to the discrete-time map

$$x_{n+2} = b_1 x_{n+1} + b_2 x_n + b_3 f(x_{n-d}), \quad (10)$$

where $b_1 = 2 - (\varepsilon_1 \Delta t)/\varepsilon_2$, $b_2 = -1 + \left(\varepsilon_1 \Delta t - (\Delta t)^2\right)/\varepsilon_2$, and $b_3 = (\Delta t)^2/\varepsilon_2$.

For each vector $\vec{X}_n = (x_{n+1}, x_n, x_{n-m})$ constructed from Eq. (9) time series we find *k* nearest neighbour vectors and estimate for them the variance σ_n^2 of the system states at the corresponding instants of time $n+2$. Then we calculate the dependence

$$D(m) = \frac{1}{N-m-3} \sum_{n=m+1}^{N-2} \sigma_n^2 \quad (11)$$

under variation of the trial delay time *m*. The location of the minimum of (11) will give us an estimation of the discrete delay time $d = \tau/\Delta t$.

The proposed methods can be used for determining an a priori unknown order of a delayed feedback system from its time series. To define the order of the time-delay system one has to recover initially its delay time under the assumption that the system is described by the first-order Eq. (1). Then, one has to recover the delay time under the assumption that the system model equation is the second-order Eq. (11) and construct the dependences (6) and (11) in the same plot. The dependence *D*(*m*) constructed under the true choice of the model equation order will lie below the dependence *D*(*m*) constructed under the false choice of the order of the model equation.

For example, let us have a time series from the second-order time-delay system (9) with quadratic nonlinear function $f(x) = \lambda - x^2$, where λ is the parameter of nonlinearity. The system parameters $\tau = 1000$, $\lambda = 1.9$, $\varepsilon_1 = 7$, and $\varepsilon_2 = 10$ correspond to chaotic oscillations. The sampling time is $\Delta t = 1$ and the number of points is $N = 10000$. Part of the time series is shown in Fig. 3(a) for the case, where a 10% dynamical noise is added into the system. Let us suppose that the order of the system model equation is unknown and first recover the delay time under the assumption that the system is governed by the first-order Eq. (1). The dependence (6) is depicted in Fig. 3(b) in black colour for $k = 10$ and $l = 10$. It has a minimum at $m = 1001$ that is slightly larger than the delay time $d = \tau/\Delta t = 1000$.

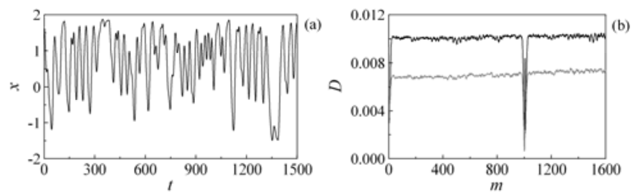

Fig. 3. (a) The time series of Eq. (9) with quadratic nonlinearity in the chaotic regime. (b) Dependences *D*(*m*) constructed under the assumption that the model equation is of the first order (black colour) and the second order (grey colour).

Let us reconstruct now the delay time assuming that the system is described by the second-order delay-differential Eq. (19). The dependence (11) is shown in Fig. 3(b) in grey colour for $k = 10$ and $l = 10$. It lies below the dependence (6) indicating that the second-order equation describes the system better than the first-order equation. The minimum of dependence (11) is observed at

m=d=1000. Thus, the delay time is recovered accurately at the true choice of the model equation order.

Then we consider the case, where a time series is gained from the first-order time-delay system (1) with quadratic nonlinear function and parameters $\tau = 1000$, $\lambda = 1.9$, and $\varepsilon = 10$ corresponding to chaotic oscillations. As well as in the considered above example, $\Delta t = 1$, $N = 10000$, and a 10% dynamical noise is added into the system.

The plot of $D(m)$ constructed under the assumption that the model equation has the form of Eq. (1) exhibits minimum at $m = d = 1000$. This plot is depicted in Fig. 4 in black colour for $k = 10$ and $l = 10$. The dependence $D(m)$ constructed under the assumption that the model equation has the form of Eq. (9) is shown in Fig. 4 in grey colour. It has a minimum at $m = 999$ and lies mainly higher than the black curve indicating that the model equation of the system has the first order.

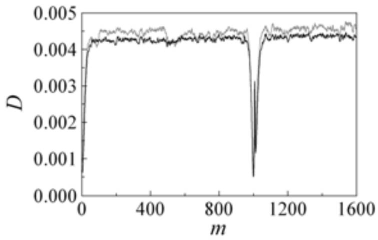

Fig. 4. Dependences $D(m)$ constructed from time series of Eq. (1) with quadratic nonlinearity under the assumption that the model equation is of the first order (black colour) and the second order (grey colour).

4. Conclusion

We have proposed the method for the reconstruction of delay time in time-delay systems from their time series. The method is based on the nearest neighbour analysis. It allows one to recover the delay times in scalar time-delay systems of different order. The method can be applied to time-delay systems with arbitrary form of nonlinear function. Moreover, the method can be used for determining an a priori unknown order of a time-delay system from its time series. The parameters of the method can be chosen within a wide range. The proposed method remains efficient under very high levels of dynamical and additive noise.

Acknowledgements

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