"Three-Shot" Technique for Stabilizing Unknown Saddle Steady States of Dynamical Systems

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Abstract—A simple model-independent proportional feedback technique for stabilizing unknown unstable steady states is described. It makes use of artificially created stable steady states to find the unknown coordinates of the inherent unstable steady states. Two different physical examples have been investigated; the mechanical pendulum, and the chaotic Duffing-Lindberg oscillator are considered both analytically and numerically.

1. Introduction

Stabilization of unstable steady states (USS) of dynamical systems is an important problem in basic science and engineering applications, when periodic or chaotic oscillations are undesirable behaviors. Conventional control methods, based on proportional feedback [1, 2] require knowledge of a mathematical model of a system or exact coordinates of the USS . However, in many real systems, especially in biology, physiology, economics, sociology, chemistry, neither the reliable models nor the exact location of the USS are *a priori* known. Moreover, the position of the USS may slowly vary with time because of external unknown and unpredictable forces. Therefore adaptive methods, automatically tracing unknown USS are needed.

A large number of adaptive methods, using either derivative feedback [3, 4], low-pass filters [5, 6, 7, 8, 9], high-pass filters [10], notch filters [11] or delayed feedback [6, 7, 12, 13, 14] have been described in literature. However, they can stabilize unstable nodes and unstable spirals only, but fail to stabilize the saddle-type USS, more specifically the USS with an odd number of real positive eigenvalues. To solve the problem of the odd number limitation Pyragas *et al.* [15, 16] proposed to use an unstable filter, that is a bold idea to fight instability with another instability. The technique has been demonstrated to stabilize saddles in several mathematical models [15, 16, 17, 18] also in the experiments with an electrochemical oscillator [15, 16] and the Duffing-Holmes-type electrical circuit [18]. This advanced method is limited, however, to dissipative dynamical systems only. It is not applicable to conservative systems. The limitation of the unstable filter method can be proved analytically using the Hurwitz stability criteria. The necessary condition for stabilizing a saddle USS is that the cut-off frequency of the unstable filter is lower than the damping coefficient of the system [16, 18]. In conservative systems damping is zero by definition. Formally, the cut-off frequency could be set negative. However, this would mean that the unstable filter should become a stable one and, therefore, inappropriate to stabilize a saddle-type USS. To get around the problem a conjoint filter, that involves unstable and stable subfilters, has been suggested and demonstrated for the Lagrange point L2 of the Sun–Earth astrodynamical system [19]. Most recent modifications of combined filter technique are described in [20, 21, 22]. The control methods proposed in [15, 18, 19, 20, 21, 22] are based on designing complex higher order controllers with several adjustable control parameters. Even linear analysis of the stability properties employs high-rank Hurwitz matrices for determining the threshold values of the feedback coefficients, while finding optimal control parameters requires numerical solution of high order characteristic equations. A simplified approach has been proposed in our previous paper [23]. Though it admits the existance of some unknown parameters, it requires, however, the explicit form of the nonlinear functions.

In this work, we suggest a multistep feedback technique. In the first and the second steps two different artificial stable steady states (SSS) are created and are exploited to find the unknown coordinates of the inherent USS. In the final third step these coordinates are used to stabilize the *a priori* unknown USS. We call this method "three–shot" technique for brevety.

2. Simple Mathematical Models

To demonstrate the idea we start with an extremely simple one-dimensional example

$$\dot{x} = F(x, p). \tag{1}$$

Here $F(x, p)$ is a nonlinear function of variable x, while p is a set of parameters. The steady states x_0 are found from an algebraic equation $F(x_0, p) = 0$. If the derrivative of $F(x, p)$ with respect to x at x_0 is positive, $F'_x(x,p)|_{x_0} > 0$, we deal with USS. The USS can be stabilized by means of proportional feedback

$$\dot{x} = F(x,p) + k(x_0 - x). \quad (2)$$

If either the structure of the function $F(x,p)$ is unknown or it contains some unknown parameter, then x_0 is also unknown. Therefore the proportional feedback cannot be applied directly. However, we demonstrate that this unknown USS can be still stabilized by multistep proportional feedback. Stabilization is achieved in three steps ("three shots"). In the first and the second steps we apply proportional feedback with an arbitrarily chosen reference points $r_{1,2}$:

$$\dot{x} = F(x,p) + k(r_{1,2} - x), \quad (3)$$

where $r_{1,2}$ are any real constants, either positive or negative (zero value is also applicable). For sufficiently large k the feedback creates artificial SSS, $x_{1,2}$, which satisfy the steady state equations $F(x_{1,2},p) + k(r_{1,2} - x_1) = 0$. Note, that the control term $k(r_{1,2} - x_1)$, in general, do not vanish, because $r_{1,2}$ are not the natural USS of the original Eq. (1). Assuming, that the chosen reference points $r_{1,2}$ are not too far from x_0, we formally linearize the nonlinear functions $F(x_{1,2},p)$ around x_0: $F(x_{1,2},p) = F(x_0,p) + F'_x(x,p)|_{x_0}(x_{1,2} - x_0)$. Here $F(x_0,p) = 0$ by definition. Then the nonlinear steady state equation read

$$F'_x(x,p)|_{x_0}(x_{1,2} - x_0) + k(r_{1,2} - x_{1,2}) = 0. \quad (4)$$

These two linear equations have two unknowns, namely $F'_x(x,p)|_{x_0}$ and x_0. Any of them or both can be easily derived. Solving of the Eqs. (4) with respect to x_0 yields:

$$x_0 = \frac{r_1 x_2 - r_2 x_1}{(r_1 + x_2) - (r_2 + x_1)}. \quad (5)$$

Eventually, we use the derived value of x_0 in the final third step of stabilization, given by Eq. (2).

As a specific mathematical example we consider Eq. (1) with $F(x,p) = ax - p$, where p is *a priori* unknown parameter. There is a single USS: $x_0 = p/a$. However, it is unknown because of p. Two preparatory steps with r_1 and r_2 give $x_1 = (kr_1 - p)/(k - a)$ and $x_2 = (kr_2 - p)/(k - a)$, respectively. Finally, the intrinsic USS x_0 is obtained from formula (5). One can check, that x_0 from (5) coincides with the expected value $x_0 = p/a$.

The technique is applicable to higher order systems as well, e.g. the second-order dynamical system

$$\dot{x} = y, \quad \dot{y} = F(x,y). \quad (6)$$

The steady states have two coordinates. One of them is trivial: $y_0 = 0$. Then x_0 is found from $F(x_0,0) = 0$. If $F'_x(x,y)|_{x_0,y_0} > 0$ or $F'_y(x,y)|_{x_0,y_0} > 0$ or both derivatives are positive, the fixed point is an USS. Depending on the structure of $F(x,y)$ and the inherent parameters the USS might be either a node, a spiral or a saddle. Any of them can be stabilized using the proportional feedback:

$$\dot{x} = y, \quad \dot{y} = F(x,y) + k_1(x_0 - x) + k_2(y_0 - y). \quad (7)$$

In the case the $F(x,y)$ is unknown, and consequently x_0 is unknown, we apply the three-step technique, similarly to the one-dimensional system. The first and the second steps are given by

$$\dot{x} = y, \quad \dot{y} = F(x,y) + k_1(r_{1,2} - x) - k_2 y. \quad (8)$$

In the second control term with coefficient k_2 we employed the fact that $y_0 = 0$. Since the y-coordinates of the artificial fixed points $y_{1,2} = 0$, the $x_{1,2}$ are found from the steady state equations: $F(x_{1,2},0) + k_1(r_{1,2} - x_{1,2}) = 0$. After linearization $F(x_{1,2},0) = F(x_0,0) + F'_x(x,y)|_{x_0,y_0}(x_{1,2} - x_0) = F'_x(x,y)|_{x_0,y_0}(x_{1,2} - x_0)$ we come to a set of two linear equations, similar to Eqs. (4) and finally to the expression for x_0, exactly the same as given by formula (5).

3. Mechanical Pendulum

Mechanical pendulum is given by

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -by - \sin x + p. \end{aligned} \quad (9)$$

Here x is the angle between the downward vertical and the rod, y is the angular velocity, b is the damping coefficient, and p is a constant, but generally unknown torque. For small torque $p < 1$, the system has two fixed points $[x_{01,02}, y_{01,02}] = [x_{01,02}, 0]$, where $x_{01} = \arcsin(p)$, $x_{02} = \pi - \arcsin(p)$. The x_{01} corresponds to the SSS (lower position of the pendulum), while the x_{02} is the x-coordinate of the saddle type USS (upper position of the pendulum). The controlled pendulum is described by

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -by - \sin x + p + k_1(x_{02} - x) - k_2 y. \end{aligned} \quad (10)$$

Linearization of Eq. (10) around x_{02} gives the characteristic equation $\lambda^2 + (b+k_2)\lambda + k_1 + \cos x_{02} = 0$. For small p the angle $x_{02} \approx \pi$, thus $\lambda_{1,2} = -(b+k_2)/2 \pm [(b+k_2)^2/4 - k_1 + 1)]^{1/2}$. The threshold value of the feedback coefficient is $k_{1th} = 1$ for which the largest eigenvalue λ_1 crosses zero from positive to negative values. The optimal value of the feedback coefficient $k_{1opt} = 1 + (b+k_2)^2/4$; the eigenvalues are both negative and equal to each other, $\lambda_1 = \lambda_2 = -(b+k_2)/2$. Further increase of k_1 makes the eigenvalues complex, but does not change their real parts. So, for higher

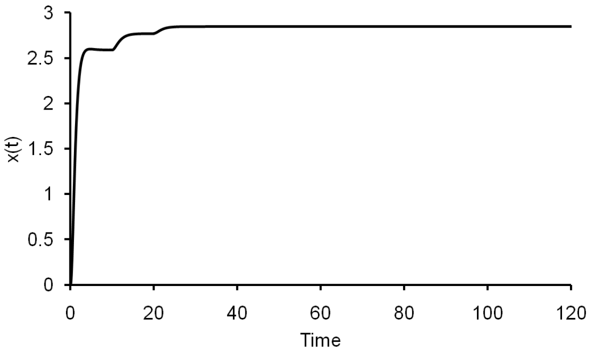

Figure 1: Stabilization of the upper position of mechanical pendulum from Eq. (10). $b = 0.1$, $p = 0.3$, $r_1 = 2.7$, $r_2 = 2.8$, $k_1 = k_2 = 2$. Initial conditions $x(0) = y(0) = 0$. The control is switched on at $t = 0$. The value stabilized in the 1st step $x_1 = 2.5867$, the value stabilized in the 2nd step $x_2 = 2.7670$, the reference point calculated from formula (5) $x_{02} = 2.8411$, the value stabilized in the 3rd step $x_3 = 2.8449$, the remaining difference in the 3rd step $\delta = x_{02} - x_3 = -0.0038$, $|\delta|/x_{02} \approx 0.1\%$, the analytical value of the UFP $x_0 = \pi - \arcsin(p) = 2.8369$.

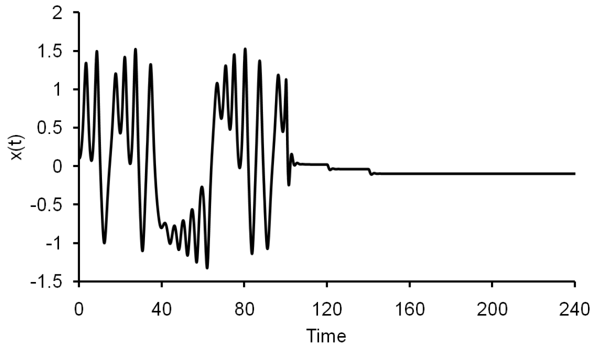

Figure 2: Stabilization of the USS of the Duffing-Lindberg oscillator from Eq. (13). $b = 0.35$, $c = 1.6$, $\omega_0 = 0.5$, $p = 0.1$, $r_1 = 0$, $r_2 = -0.05$, $k_1 = 6$, $k_2 = 2$. Initial conditions $x(0) = 0.1$, $y(0) = z(0) = 0$. The control is switched on at $t = 100$. The value stabilized in the 1st step $x_1 = 0.0200$, the value stabilized in the 2nd step $x_2 = -0.0400$, the reference point calculated from formula (5) $x_{02} = -0.1$, the value stabilized in the 3rd step $x_3 = -0.0998$, the remaining difference in the 3rd step $\delta = x_{02} - x_3 = -0.0002$, $|\delta|/|x_{02}| \approx 0.2\%$, the analytical value of the USS from (12) $x_{02} = -0.1010$.

feedback coefficients the convergence rate saturates with k_1 and is fully determined by $(b + k_2)$. In the case of weak damping ($b \ll 1$) a reasonable pair of the feedback coefficients is $k_1 = 2$ and $k_2 = 2$, yielding $\mathrm{Re}\lambda_{1,2} \approx -1$. Results are shown in Fig. 1.

4. Duffing-Lindberg Chaotic Oscillator

Duffing–Lindberg oscillator is described by [24]:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 + by - cz + p, \\ \dot{z} &= \omega_0(y - z). \end{aligned} \tag{11}$$

For $|p| < 2/\sqrt{27}$ the system has three fixed points $[x_0, y_0, z_0] = [x_0, 0, 0]$. The x-coordinates are found from a cubic steady state equation $x_0^3 - x_0 - p = 0$. For $p = 0$ the solution is simple: $x_{01} = -1$, $x_{02} = 0$, $x_{03} = 1$. For non-zero p the expressions are:

$$\begin{aligned} x_{01} &= -h\cos\frac{\pi-\theta}{3},\ x_{02} = -h\cos\frac{\pi+\theta}{3}, \\ x_{03} &= h\cos\frac{\theta}{3},\ h = \frac{2}{\sqrt{3}},\ \theta = \arccos\frac{3p}{h} \end{aligned} \tag{12}$$

All the fixed points are unstable. The side fixed points, x_{01} and x_{03} are either unstable nodes or unstable spirals. The most complicated is the middle one, x_{02} in the sense that it is a saddle with one positive eigenvalue. Similarly to the previous examples we apply proportional feedback in the form of $k_1(x_0 - x) - k_2 y$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 + by - cz + p + k_1(x_{02} - x) - k_2 y, \\ \dot{z} &= \omega_0(y - z). \end{aligned} \tag{13}$$

Here we note, that in equation for variable y it is possible to use one more feedback term, namely $-k_3 z$. However, two terms are sufficient for stabilization. Linearization of Eqs. (13) around x_{02} leads to a cubic characteristic equation $\lambda^3 + a_3\lambda^2 + a_2\lambda + a_1 = 0$ with

$$a_1 = \omega_0(k_1 - 1), \tag{14}$$

$$a_2 = k_1 + \omega_0(k_2 + c - b) - 1, \tag{15}$$

$$a_3 = k_2 + \omega_0 - b. \tag{16}$$

When deriving expressions (14–16) we assumed for simplicity that $3x_{02}^2 \ll 1$. The third-order system (13) is stable, if the eigenvalues $\mathrm{Re}\lambda_{1,2,3}$ of the characteristic equation are all negative. $\mathrm{Re}\lambda_{1,2,3} < 0$, if the all following inequalities are fulfilled:

$$a_1 > 0, \quad a_3 a_2 - a_1 > 0, \quad a_3 > 0. \tag{17}$$

The $a_1 > 0$, if $k_1 > 1$. Once $k_1 > 1$, the second inequality is easily fulfilled for the given parameters c and b (even for $k_2 = 0$). Finally, $a_3 > 0$ holds for the given parameters ω_0 and b (even for $k_2 = 0$). However, $k_2 > 0$ makes the transients shorter. Numerical results are presented in Fig. 2.

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References

[1] B. C. Kuo, *Automatic Control Systems*, Prentice Hall, Englewood Cliffs, New Jersey, 1995.

[2] K. Ogata, *Modern Control Engineering*, Prentice Hall, Englewood Cliffs, New Jersey, 2010.

[3] S. Bielawski, M. Bouazaoui, D. Derozier, P. Glorieux, "Stabilization and characterization of unstable steady states in a laser," *Phys. Rev. A*, vol.47, pp.3276–3279, 1993.

[4] P. Parmananda, M. A. Rhode, G. A. Johnson, R. W. Rollins, H. D. Dewald, A. J. Markworth, "Stabilization of unstable steady state in an electrochemical system using derivative control," *Phys. Rev. E*, vol.49, pp.5007–5013, 1994.

[5] N. F. Rulkov, L. S. Tsimring, H. D. I. Abarbanel, "Tracking unstable orbits in chaos using dissipative feedback control," *Phys. Rev. E*, vol.50, pp.314–324, 1994.

[6] A. Namajūnas, K. Pyragas, A. Tamaševičius, "Stabilization of an unstable steady state in a Mackey-Glass system," *Phys. Lett. A*, vol.204, pp.255–262, 1995.

[7] A. Namajūnas, K. Pyragas, A. Tamaševičius, "Techniques for modeling and controlling the Mackey-Glass system," *Int. J. Bifurcation Chaos*, vol.7, pp.957–962, 1997.

[8] A. Schenk zu Schweinsberg, U. Dressler, "Characterization and stabilization of the unstable fixed points of a frequency doubled Nd:YAG laser," *Phys. Rev. E*, vol.63, 056210, 2001.

[9] H. Huijberts, "Linear controllers for the stabilization of unknown steady states of chaotic systems," *IEEE Trans. Circuits Syst. – I: Regular Papers*, vol.53, pp.2246–2254, 2006.

[10] M. Ciofini, A. Labate, R. Meucci, M. Galanti, "Stabilization of unstable fixed points in the dynamics of a laser with feedback," *Phys. Rev. E*, vol.60, pp.398–402, 1999.

[11] A. Ahlborn, U. Parlitz, "Chaos control using notch filter feeedbck," *Phys. Rev. Lett.*, vol.96, 034102, 2006.

[12] K. Pyragas, "Control of chaos via extended delay feedback," *Phys. Lett. A*, vol.206, pp.323–330, 1995.

[13] P. Hövel, E. Schöll, "Control of unstable steady states by time delayed feedback methods," *Phys. Rev. E*, vol.72, 046203, 2005.

[14] A. Gjurchinovski, T. Jungling, V. Urumov, E. Schöll, "Delayed feedback control of unstable steady states with high-frequency modulation of delay," *Phys. Rev. E*, vol.88, 032912, 2013.

[15] K. Pyragas, V. Pyragas, I. Z. Kiss, J. L. Hudson, "Stabilizing and tracking unknown steady states of dynamical systems," *Phys. Rev. Lett.*, vol.89, 244103, 2002.

[16] K. Pyragas, V. Pyragas, I. Z. Kiss, J. L. Hudson, "Adaptive control of unknown unstable steady states of dynamical systems," *Phys. Rev. E*, vol.70,026215, 2004.

[17] D. J. Braun, "Adaptive steady-state stabilization for nonlinear dynamical systems," *Phys. Rev. E*, vol.78, 016213, 2008.

[18] A. Tamaševičius, E. Tamaševičiūtė, G. Mykolaitis, S. Bumelienė, "Switching from stable to unknown unstable steady states of dynamical systems," *Phys. Rev. E*, vol.78, 026205, 2008.

[19] A. Tamaševičius, E. Tamaševičiūtė, G. Mykolaitis, S. Bumelienė, R. Kirvaitis, "Stabilization of saddle steady states of conservative and weakly damped dissipative dynamical systems," *Phys. Rev. E*, vol.82, 026205, 2010.

[20] A. Tamaševičius, E. Tamaševičiūtė, G. Mykolaitis, S. Bumelienė, "Enhanced control of saddle steady states of dynamical systems," *Phys. Rev. E*, vol.88, 032904, 2013.

[21] E. Tamaševičiūtė, G. Mykolaitis, S. Bumelienė, A. Tamaševičius, "Stabilizing saddles," *Phys. Rev. E*, vol.88, 060901(R), 2013.

[22] E. Tamaševičiūtė, G. Mykolaitis, S. Bumelienė, A. Tamaševičius, "Tracking and controlling unstable steady states of dynamical systems," *Commun. Nonlinear Sci. Numer. Simulat.*, vol.19, pp.649–655, 2014.

[23] E. Tamaševičiūtė, A. Tamaševičius, "Stabilizing uncertain steady states of some dynamical systems by means of proportional feedback," *Nonlinear Anal. Model. Control*, vol.18, pp.86–98, 2013.

[24] E. Lindberg, E. Tamaševičiūtė, G. Mykolaitis, S. Bumelienė, T. Pyragienė, A. Tamaševičius, R. Kirvaitis, "Autonomous third-order Duffing-Holmes type chaotic oscillator," *Proc. Eur. Conf. on Circuit Theory and Design*, 23-27 August, 2009, Antalya, Turkey, pp.663-666, IEEE, New York, 2009.