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Model Predictive Control of a powertrain system with backlash

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Abstract—
In this paper we apply a switched Model Predictive Control strategy for the speed regulation of a powertrain system with backlash, composed by a motor and a load connected through an elastic shaft and a gear. We show that the proposed control strategy allows for a fast tracking, even in the presence of load torque, by completely suppressing rebounds on the gear teeth, which are instead evident if simpler (e.g., proportional-integral) control strategies are adopted.

1. Introduction

Permanent Magnet Synchronous Motors are widely used in industrial processes due to their high power density and the relatively low control complexity. Among the possible application fields of these devices, there are robot manipulators, numeric control machines, rolling-mills and paper machines. Direct drive systems with high stiffness are usually adopted when high performance is required [1]. In some applications, however, geometrical constraints impose the use of transmission elements such as shafts and gears. The elasticity of the shaft and the backlash introduced by the gear provoke notable effects in the system, that have to be compensated by the controller in order to achieve fast and smooth responses, without rebounds in the gear teeth which may cause premature component wear.

Traditionally used controllers in industrial field, such as proportional-integral (PI) controllers, are often not able to handle these aspects, resulting in low quality dynamic responses of the system and sometimes in unstable behaviors. The control of powertrain systems with backlash has been widely studied in recent literature [2, 3, 4]. In [4], in particular, the system is modeled as a piecewise-affine (PWA) system and is controlled with Model Predictive Control (MPC) technique.

MPC is a popular technique for the control of constrained linear systems. In its classical formulation, the computation of the control function requires the online solution of an optimization problem, which is a quite demanding task. In [5] the computation was brought offline thanks to multi-parametric programming and an explicit solution was obtained, which results being a PWA function of the system states. MPC formulation can be extended for the control of PWA systems and the resulting solution is again a PWA function of the state, usually defined over a large number of partitions. A suitable strategy to reduce drastically the number of partitions (at the cost of a loss of performance) is the switched MPC approach, successfully applied for instance in [6]. This approach consists in solving different MPC optimization problems, one for each affine system dynamics. This allows to define different cost functions and constraints for each MPC controller.

In this work we exploit switched MPC for the speed regulation of a powertrain system with backlash. The resulting control function is a PWA function defined over a very small number of regions with respect to the solution proposed in [4]. We also provide a comparative study with a classical PI, in order to show the advantages of the proposed control algorithm.

This work is intended as a theoretical validation of a control strategy, indeed we do not take into account errors due to state estimation, model uncertainties, delays and quantization effects. Moreover we completely exclude the electrical dynamics from our analysis.

2. Notation

Subscript k is used to indicate the discrete-time instant, i.e., $x_k = x(k\tau)$, being τ the sampling period. If x is the state of a dynamical system, the notation $x_{k+i|k}$ represents the predicted state at time $(k+i)\tau$ starting from state x_k. Given a matrix M, M_{ij} is the element in i-th row and j-th column. $M > 0$ indicates the positive definiteness of the matrix, and $M \geq 0$ the positive semi-definiteness. All inequalities involving arrays are to be intended componentwise. $\mathbb{1}(t)$ is the step function, i.e., $\mathbb{1}(t) = 0$ if $t < 0$ and $\mathbb{1}(t) = 1$ if $t \geq 0$.

3. Model of the system

We consider a mechanical system composed by an AC motor (characterized by momentum of inertia J_m and viscous friction b_m) connected to a rotating load (with momentum of inertia J_l and viscous friction b_l) by an elastic shaft equipped with a gear (see Fig. 1 for a schematic representation). The motor generates a driving torque T_m, and the shaft (with elastic constant c and damping factor b) can in turn produce a torque T_s. A disturbance torque T_l acting on the load is also considered.

The presence of the gear introduces the phenomenon of backlash, which causes the motor and the load to be uncou-

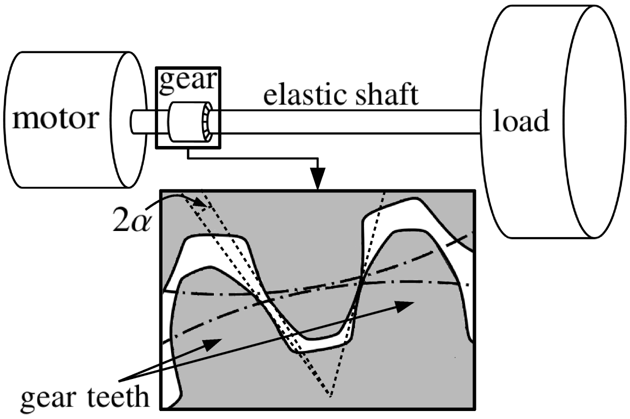

Figure 1: Schematic representation of the powertrain system with backlash.

pled for a small amount of time when the motor goes from braking to acceleration and vice versa. The backlash gap angle is denoted as 2α (see Fig. 1).

We briefly recall the modeling strategy for this system as proposed in [4]. In the next section we will show some simplifications to this model, necessary to design a simpler controller. The system can operate in two distinct modes: *contact* when motor and load are coupled and *backlash* when they are uncoupled. By defining θ_b the backlash angle, we can state that the system is in *backlash* mode when one of the following conditions holds:

$$|\theta_b| < \alpha \tag{1}$$

$$\theta_b = \alpha \text{ and } \Delta\dot{\theta} + \frac{c}{b}(\Delta\theta - \theta_b) < 0 \tag{2}$$

$$\theta_b = -\alpha \text{ and } \Delta\dot{\theta} + \frac{c}{b}(\Delta\theta - \theta_b) > 0 \tag{3}$$

Inequality (1) indicates that motor and load are uncoupled. (2) and (3) are limit conditions, in which motor and load are in contact but they are going to uncouple at the next time instant.

In each of the working modes the system dynamics is affine, therefore the overall system can be modeled with a PWA model:

$$\dot{x} = \begin{cases} A_{co}x + Bu + fw & (contact) \\ A_{bl}x + Bu + fw & (backlash) \end{cases} \tag{4}$$

being $x = [\omega_m\ \omega_l\ \theta_m\ \theta_l\ \theta_b]^T$, $u = T_m$ and $w = T_l$ the system state, input and disturbance, respectively; θ_m, ω_m, θ_l and ω_l are the angles and angular speeds of the motor and load, respectively. The system matrices are defined as follows:

$$A_{co} = \begin{bmatrix} -\frac{b_m+b}{J_m} & \frac{b}{J_m} & -\frac{c}{J_m} & \frac{c}{J_m} & \frac{c}{J_m} \\ \frac{b}{J_l} & -\frac{b_l+b}{J_l} & \frac{c}{J_l} & -\frac{c}{J_l} & -\frac{c}{J_l} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

$$A_{bl} = \begin{bmatrix} -\frac{b_m}{J_m} & 0 & 0 & 0 & 0 \\ 0 & -\frac{b_l}{J_l} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & \frac{c}{b} & -\frac{c}{b} & -\frac{c}{b} \end{bmatrix} \tag{6}$$

$$B = \begin{bmatrix} \frac{1}{J_m} & 0 & 0 & 0 & 0 \end{bmatrix}^T \tag{7}$$

$$f = \begin{bmatrix} 0 & \frac{1}{J_l} & 0 & 0 & 0 \end{bmatrix}^T \tag{8}$$

In this paper we assume that the system state and the disturbance (x and w) are completely measurable. In real applications, usually, only the motor angle is measurable and, therefore, the design of an observer would be necessary.

Table 1 shows the values of all system parameters, related to a AC motor providing the driving torque and a DC motor acting as load.

Name	Value	Unit
J_m	$1 \cdot 10^{-3}$	$kg\ m^2\ rad^{-1}$
J_l	$4 \cdot 10^{-3}$	$kg\ m^2\ rad^{-1}$
b_m	$1 \cdot 10^{-4}$	$N\ m\ s\ rad^{-1}$
b_l	$2 \cdot 10^{-4}$	$N\ m\ s\ rad^{-1}$
c	$2.0213 \cdot 10^{4}$	$N\ m\ rad^{-1}$
b	$8.042 \cdot 10^{-1}$	$N\ m\ s\ rad^{-1}$
α	$8.7 \cdot 10^{-3}$	rad

Table 1: Model parameters.

4. Model Predictive Control

4.1. Description

Consider a discrete-time affine system in the form:

$$x_{k+1} = Ax_k + Bu_k + f \tag{9}$$

subject to constraints:

$$u_{min} \le u_k \le u_{max} \tag{10}$$

$$x_{min} \le x_k \le x_{max} \tag{11}$$

where x_k and u_k denote the system state and input. The MPC technique provides a control function $u_k = u(x_k)$ for the regulation of the system state to a reference state x_{ref} by solving the following optimization problem [5]:

$$\min_{u_k,\dots,u_{k+N_u-1}} \quad (x_{k+N} - x_{ref})^T P (x_{k+N} - x_{ref}) + \tag{12}$$

$$+ \sum_{i=0}^{N-1} \left\{ (x_{k+i|k} - x_{ref})^T Q (x_{k+i|k} - x_{ref}) + u_{k+i}^T R u_{k+i} \right\}$$

$$\text{s.t.} \quad x_{k+i+1|k} = Ax_{k+i|k} + Bu_{k+i} + f \tag{13}$$

$$u_{k+i} = 0, \quad i \ge N_u \tag{14}$$

$$u_{min} \le u_{k+i} \le u_{max}, \quad i = 0, \dots, N_c \tag{15}$$

$$x_{min} \le x_{k+i|k} \le x_{max}, \quad i = 1, \dots, N_c \tag{16}$$

where N, N_u and N_c are the prediction horizon, control horizon and constraints horizon, respectively; $P = P^T \geq 0$, $Q = Q^T \geq 0$, $R = R^T > 0$ are matrices of a proper size. The above problem can be solved explicitly [5] by obtaining a control law u_k which results being a PWA function of the system state x_k defined over generic convex polytopes. In the solution of the optimization problem, only u_k is considered; all other functions $u_{k+1}, \ldots, u_{k+N_u-1}$ are discarded.

This formulation can be extended to hybrid systems (including PWA systems like (4)). The resulting control law is again a PWA function of the system state but is usually defined over a large number of polytopes. In order to reduce the complexity of the controller, we employed a switched approach [6], i.e., we solved separate MPC problems in the form (12), by considering only one affine dynamics at once.

4.2. Model approximation

In order to design a simpler MPC controller, we make some modifications to model (4). Since the dynamics of the whole system depends on $\Delta\theta = \theta_m - \theta_l$, rather than on the two angles separately, we can consider $\Delta\theta$ as a state variable instead of θ_m and θ_l. Furthermore, we notice that state variable θ_b can be removed; θ_b can be assumed (with good approximation degree) equal to: $\Delta\theta$ when the system is in *backlash mode*, α when $\Delta\theta \geq \alpha$, and $-\alpha$ when $\Delta\theta \leq -\alpha$. Moreover, conditions (1)-(3) can be simplified as follows:

$$|\Delta\theta| < \alpha \tag{17}$$

After these simplifications, we discretize the system with zero-order hold method, since MPC technique requires a discrete-time representation of the system. Finally, instead of u_k, we consider as an input to the system the increment $\Delta u_k = u_k - u_{k-1}$. This implies that u_{k-1} must be considered as a state variable, in order to perform tracking.

4.3. Control design

4.3.1. Contact

The objective of the controller is to regulate the angular speed of the load (ω_l) to a given reference (ω_{ref}), by imposing an a proper driving torque (T_m) fulfilling input and state constraints. This task can be accomplished only if the system is in *contact* mode, since in *backlash* the motor and the load are uncoupled.

As stated in section 4.2, due to the removal of state variable θ_b, we can distinguish between two different contact dynamics: the first one (*positive contact*) applies when $\Delta\theta \geq \alpha$, the second one (*negative contact*) is valid when $\Delta\theta \leq -\alpha$. Since the objective of the control is to bring ω_l to ω_{ref}, we can replace state variable ω_l with the tracking error $e_l = \omega_l - \omega_{ref}$, to be regulated to 0. Also the reference speed ω_{ref} and the load torque T_l must be included in the state vector, in order to formulate each contact dynamics as in equation (9). These two values are assumed to be constant in prediction phase. The state vector is therefore defined as $x_k = [\omega_{m,k}\ e_{l,k}\ \Delta\theta_k\ u_{k-1}\ \omega_{ref,k}\ T_{l,k}]^T$ and the reference state as $x_{ref} = [0\ 0\ 0\ 0\ 0\ 0]^T$.

We designed two different MPC controllers for the *contact* working condition: the control law Δu_k^{PC} for *positive contact* and the function Δu_k^{NC} for *negative contact*. The control parameters for both controllers are: $\tau = 250\mu s$, $N = 70$, $N_u = 2$, $N_c = 1$, $Q_{22} = 1000$,[1] $P = Q$, $R = 1$. The bounds for system states and inputs are: $x_{max} = -x_{min} = [100\frac{rad}{s}\ 200\frac{rad}{s}\ 2\alpha\ 10Nm\ 100\frac{rad}{s}\ 10Nm]^T$ and $u_{max} = -u_{min} = 20Nm$.

4.3.2. Backlash

As stated before, when the system is in *backlash* mode, it is not possible to control the load speed by imposing a motor torque. Therefore, the objective in this case is to connect the motor to the load as quickly, soon as possible, so that the driving torque can be transmitted. This means bringing the angular displacement $\Delta\theta$ to either α (*positive backlash*) or $-\alpha$ (*negative backlash*), according to the sign of the tracking error e_l. Indeed, $e_l > 0$ means that the load speed is greater than the reference speed, and therefore it is necessary to bring the angular displacement to $-\alpha$ in order to brake the load. Conversely, $\Delta\theta$ must be brought to α in order to accelerate the load. Moreover, to avoid rebounds on the gear teeth, it is important that motor and load connect with the same angular speed, i.e., $\omega_m = \omega_l$. State and input constraints must be imposed too.

Also in this case we designed two different MPC controllers: the control law Δu_k^{PB} for *positive backlash* and the function Δu_k^{NB} for *negative backlash*. The state vector is defined as $x_k = [\omega_{m,k}\ \omega_{l,k}\ \Delta\theta_k\ u_{k-1}\ T_{l,k}]^T$ and the reference state as $x_{ref} = [0\ 0\ \alpha\ 0\ 0]^T$ for *positive backlash* and $x_{ref} = [0\ 0\ -\alpha\ 0\ 0]^T$ for *negative backlash*. Notice that ω_{ref} is not included in the state vector since it is not necessary for the control objective.

The control parameters are: $\tau = 250\mu s$, $N = 12$, $N_u = 2$, $N_c = 1$, $Q_{11} = Q_{22} = 100$, $Q_{12} = Q_{21} = -100$, $Q_{33} = 10$,[1] $P = Q$, $R = 0.1$. The bounds for system states and inputs are: $x_{max} = -x_{min} = [100\frac{rad}{s}\ 100\frac{rad}{s}\ 2\alpha\ 10Nm\ 10Nm]^T$ and $u_{max} = -u_{min} = 20Nm$.

In conclusion we designed four distinct MPC controllers, one for each working mode (*positive contact*, *negative contact*, *positive backlash*, *negative backlash*). According to the value of $\Delta\theta$ and e_l, the appropriate controller is chosen, as summarized in Fig. 2.

5. Results

We designed the four MPC control laws Δu_k^{PC}, Δu_k^{NC}, Δu_k^{PB} and Δu_k^{NB} by using MOBY-DIC Toolbox [7] interfaced with Multi Parametric Toolbox [8]. These control laws are PWA functions defined over 9, 9, 15 and 15 polytopes, respectively. For the sake of comparison, we also de-

[1] All remaining elements of matrix Q are set to 0.

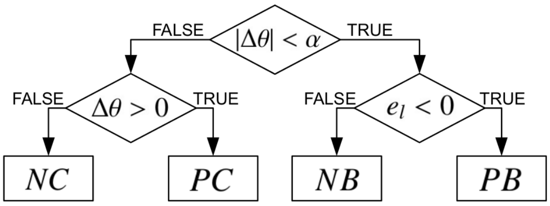

Figure 2: Flowchart showing the switching conditions for choosing one of the four controllers.

signed a discrete-time PI controller $u_k^{PI} = K_p(\omega_{ref,k} - \omega_{l,k}) + K_i\tau\sum_{j=0}^{k}(\omega_{ref,k} - \omega_{l,k})$ with $K_p = 2.5$ and $K_i = 75$.[2] The performances of MPC and PI controllers are compared on a benchmark scenario where we impose a piecewise-constant reference speed and load torque, in particular: $\omega_{ref}(t) = -100[\mathbb{1}(t) - \mathbb{1}(t-t_1)] + 100\ \mathbb{1}(t-t_1)$ and $T_l(t) = -9\mathbb{1}(t-t_2)$ with $t_1 = 0.075s$ and $t_2 = 0.2s$. The Simulink simulation results of the closed-loop system, composed by the continuous-time plant (4) and the MPC and PI controllers, are shown in Fig. 3.

It can be noticed that both controllers allow correctly tracking the reference speed profile also in the presence of a high load torque, as visible in the upper panel of Fig. 3. The response is slightly faster when the PI controller is applied, but the MPC controller completely removes the rebounds on the gear teeth, as highlighted in the zoomed rectangles in Fig. 3. In correspondence of the switching instants, indeed, it can be noticed that the MPC control function differs significantly from the PI one (see bottom panel).

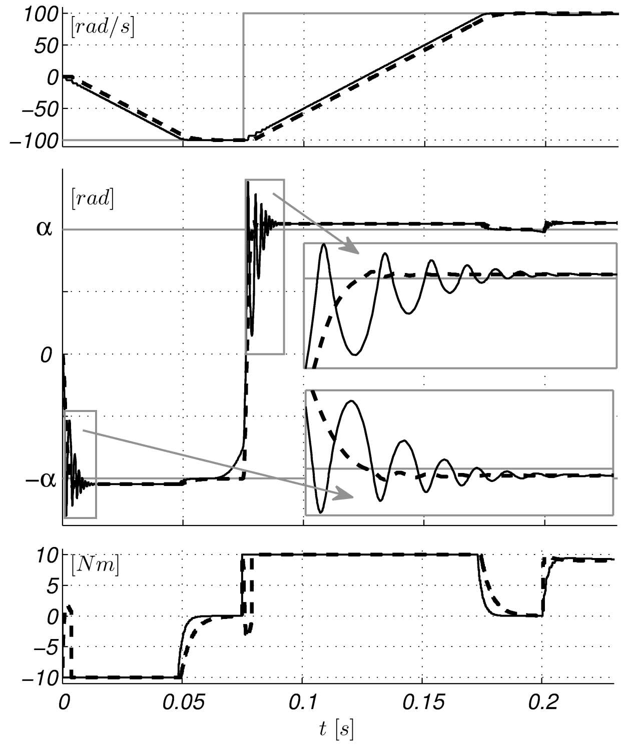

Figure 3: Closed-loop system simulation results. Top panel: ω_l (MPC: dashed line; PI: solid black line) and ω_{ref} (solid gray line); middle panel: $\Delta\theta$ (MPC: dashed line; PI: solid black line) and $\pm\alpha$ (solid gray line); bottom panel: T_m (MPC: dashed line; PI: solid black line).

6. Conclusions

We designed a switched MPC control system for the speed regulation of a rotating load connected to a driving motor through an elastic shaft with a gear. The control system is composed by four different MPC controllers with a very simple structure. The proposed controller outperforms a more traditional PI controller, since it allows completely removing the rebounds on the gear teeth, without losing convergence speed.

References

[1] M. Calvini, A. Formentini, G. Maragliano, and M. Marchesoni. Self-commissioning of direct drive systems. In *Proc. SPEEDAM*, pages 1348–1353, 2012.

[2] M. B. Khan, F. M. Malik, and K. Munawar. Switched hybrid speed control of elastic systems with backlash. In *Proc. IEEM*, pages 1641–1644, 2010.

[3] J. Huang, C. Li, and B.Y. Duan. Adaptive backstepping integral control of servo systems with backlash. *Proc. IME, Part I*, 225(8):1149–1162, 2011.

[4] P. Rostalski, T. Besselmann, M. Barić, F. Van Belzen, and M. Morari. A hybrid approach to modelling, control and state estimation of mechanical systems with backlash. *Int. J. of Control*, 80(11):1729–1740, 2007.

[5] A. Bemporad, M. Morari, V. Dua, and E. N. Pistikopoulos. The explicit linear quadratic regulator for constrained systems. *Automatica*, 38(1):3–20, 2002.

[6] S. Di Cairano and H. Tsengz. Driver-assist steering by active front steering and differential braking: design, implementation and experimental evaluation of a switched model predictive control approach. In *Proc. CDC*, pages 2886–2891, 2010.

[7] A. Oliveri, D. Barcelli, A. Bemporad, B. A. G. Genuit, W. P. M. H. Heemels, T. Poggi, M. Rubagotti, and M. Storace. MOBY-DIC: A MATLAB toolbox for circuit-oriented design of explicit MPC. In *Proc. NMPC*, volume 27, 2012.

[8] M. Kvasnica, P. Grieder, M. Baotić, and M. Morari. Multi-parametric toolbox (MPT). In *Hybrid systems: comput. and contr.*, pages 448–462. Springer, 2004.

[2] These values are set heuristically in order to achieve the fastest response without significant speed oscillations and offset.