Bifurcation of hyperchaos in a manifold piecewise linear system

Kentaro Yanagisawa† and Tadashi Tsubone‡

†Nagaoka University of Technology
1603-1, Kamitomioka, Nagaoka, Niigata, 940-2188, Japan
Email: s113231@stn.nagaokaut.ac.jp
‡Nagaoka University of Technology
Email: tsubone@vos.nagaokaut.ac.jp

Abstract—We consider bifurcation of hyperchaos in a four-dimensional manifold piecewise linear system with hysteresis characteristics. The system dynamics is reduced to a two dimensional piecewise linear return map. By using the return map, we have discussed the generation of hyperchaos in previous works. In this paper, we focus on bifurcation phenomena which are seen as changes of invariant sets of hyperchaos attractor of 2-D return map. We derive the bifurcation sets and confirm some bifurcation of hyperchaos in laboratory measurements.

1. Introduction

Hyperchaos was introduced by Rössler in 1979 [1]. It is a high dimensional chaos that has more than one positive Lyapnov exponent on 4 or more dimensional state space in the case of autonomous and continuous time systems. Many systems which exhibit hyperchaos have been studied, for example, Lorenz system [2], Chua's circuit-based high-dimensional system [3] and four-dimensional (4-D) manifold piecewise linear system [4]. In this study, we consider the 4-D manifold piecewise linear system. The system has three continuous state variables and one discrete variable which takes two values, 1 or -1. The system dynamics is described by two 3-D linear equations connected to each other by the switching which is represented by the discrete state. The system dynamics can be reduced to 2-D piecewise linear return map. The return map enable us to analysis of bifurcation and system stability. Hyperchaos observed in 2-D discrete time system is characterized by two positive Lyapnov exponents. There are many important works of nonlinear circuit analysis based on its 1-D return map [5]. Some interesting works based on 2-D return map has also been done relative to study of symple switched dynamical systems [6][7]. On the other hand, many works of 2-D discrete time nonlinear dynamical systems have been studied, for example, Hénon map [8], Lozi map [9] and Lozi map with hysteresis characteristics [10]. In this paper, we consider a hysteresis return map derived from a 4-D manifold piecewise linear system. The return map exhibits bifurcation of hyperchaos such as generation of Islands [11] and separation of invariant sets of attractor of the return map. We show bifurcation sets in parameter space and confirm typical bifurcation phenomena by an experimental circuit.

2. Manifold piecewise linear system with hysteresis characteristic

We introduce a 4-D manifold piecewise linear system with hysteresis characteristic. The system dynamics is described as follows.

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}=\begin{bmatrix}0&1&0\\-1&2\delta&0\\0&0&\lambda\end{bmatrix}\left[\begin{bmatrix}x\\y\\z\end{bmatrix}-\begin{bmatrix}1\\0\\-\frac{K}{\lambda}\end{bmatrix}u\right], \tag{1}$$

where "·" is differential by normalization time τ, and δ, λ and K are parameters which satisfy the following conditions.

$$0<\delta<1, 0<\lambda<1, K>0. \tag{2}$$

x, y and z are continuous variables and u is a binary variable in order to describe the switching mechanism. We prepare two 3-D state spaces corresponding to u :

$$\begin{aligned}S_+ &= \{(x,y,z,u)|u=1\},\\ S_- &= \{(x,y,z,u)|u=-1\}.\end{aligned} \tag{3}$$

An example of system behavior on state spaces is shown in Fig. 1. The trajectories on subspace S_+ and S_- behave as expanding oscillation on each space. If a trajectory on S_+ satisfy $z > x+D$ and $y = 0$, then u changes 1 to -1 and the trajectory jumps to same point on S_-. In like manner, if a trajectory on S_- satisfy $z < x-D$ and $y = 0$, then u changes -1 to 1 and the trajectory jumps to same point on S_+. The system repeats this behavior. Note that the parameter D is positive real number. The system in previous work [4] corresponds to the case of $D = 0$. The system dynamics is characterized by 4 parameters (δ, λ, K, D). The piecewise exact solutions are represented as follows.

$$\begin{bmatrix}x(\tau)\mp 1\\ y(\tau)\\ z(\tau)\pm\frac{K}{\lambda}\end{bmatrix}=$$
$$\frac{1}{\omega}e^{\delta\tau}\begin{bmatrix}\cos(\phi+\omega\tau)&\sin\omega\tau&0\\-\sin\omega\tau&\cos(\phi-\omega\tau)&0\\0&0&\omega e^{(\lambda-\delta)\tau}\end{bmatrix}\begin{bmatrix}x(0)\mp 1\\ y(0)\\ z(0)\pm\frac{K}{\lambda}\end{bmatrix}, \tag{4}$$

where $\phi = \tan^{-1}\frac{\delta}{\omega}$. Figure 2 shows a typical hyperchaotic attractor on 2-D projection. The block

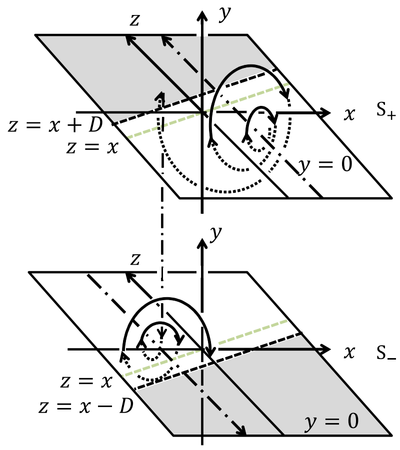


Figure 1: A typical trajectory on phase space

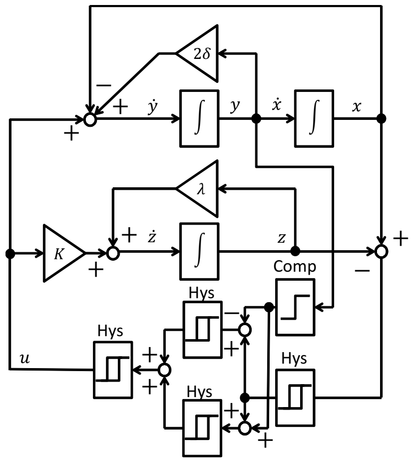
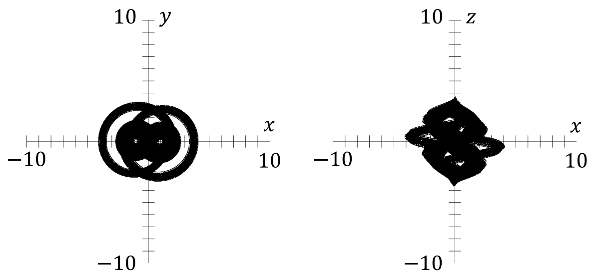
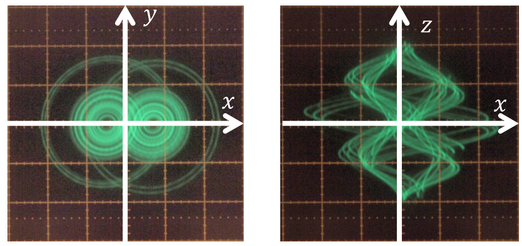


Figure 3: The block diagram of 4-D manifold piecewise linear system



(a) $x-y$ plane (b) $x-z$ plane

Figure 2: The typical hyperchaos attractor ($D = 0, \delta = 0.03, \lambda = 0.03, K = 0.5$)



(a) $x-y$ plane (b) $x-z$ plane

Figure 4: Hyperchaos attractor of experimental circuit

diagram of system is shown in Fig. 3. This circuit consists of integrators, linear amplifiers and comparators. The integrators and amplifiers are realized by using operational amplifier TL074. The comparators are implemented by LM339. There are two types of comparators, one has a Signum-like characteristic and the other has hysteresis. A hyperchaos attractor observed from the experimental circuit is shown in Fig. 4. Circuit parameters are set to corresponding values to the simulation result as shown in Fig. 2.

3. Hysteresis return map

We derive 2-D return map in order to analyze bifurcation phenomena. We define a set $P = \{(x, y, z, u) | y = 0\}$ on phase space as Poincaré section. Note that a trajectory stating from any point on P at $\tau = 0$ must return to P at $\tau = \pi/\omega$. Therefore, we can define a return map $\mathbf{F}$ from P to itself.

$$\mathbf{F} : P \to P, \ (x_n, z_n, u_n) \mapsto (x_{n+1}, z_{n+1}, u_{n+1}). \tag{5}$$

The point on P is determined as 2-D coordinate (x, z) and u, and let n-th point be represented by (x_n, z_n, u_n). The map $\mathbf{F}$ is described explicitly as follows by using piecewise exact solutions (7).

$$\begin{bmatrix} x_{n+1} \\ z_{n+1} \\ u_{n+1} \end{bmatrix} = \mathbf{F} \begin{bmatrix} x_n \\ z_n \\ u_n \end{bmatrix} = \begin{cases} \mathbf{F}_+ \begin{bmatrix} x_n \\ z_n \\ u_n \end{bmatrix} \text{for} & \begin{array}{l} z_n < x_n - D \\ \text{or } (z_n < x_n + D \\ \text{and } u_n = 1), \end{array} \\ \mathbf{F}_- \begin{bmatrix} x_n \\ z_n \\ u_n \end{bmatrix} & \text{otherwise,} \end{cases}$$

$$\begin{aligned} \mathbf{F}_+ \begin{bmatrix} x_n \\ z_n \\ u_n \end{bmatrix} &= \begin{bmatrix} f_+(x_n) \\ g_+(z_n) \\ 1 \end{bmatrix}, \\ \mathbf{F}_- \begin{bmatrix} x_n \\ z_n \\ u_n \end{bmatrix} &= \begin{bmatrix} f_-(x_n) \\ g_-(z_n) \\ -1 \end{bmatrix}, \end{aligned} \tag{6}$$

$$\begin{cases} f_+(x_n) = -e^{\frac{\delta\pi}{\omega}} x_n + (e^{\frac{\delta\pi}{\omega}} + 1), \\ g_+(z_n) = e^{\frac{\lambda\pi}{\omega}} z_n + (e^{\frac{\lambda\pi}{\omega}} - 1)K/\lambda, \\ f_-(x_n) = -e^{\frac{\delta\pi}{\omega}} x_n - (e^{\frac{\delta\pi}{\omega}} + 1), \\ g_-(z_n) = e^{\frac{\lambda\pi}{\omega}} z_n - (e^{\frac{\lambda\pi}{\omega}} - 1)K/\lambda. \end{cases} \tag{7}$$

4. Bifurcation analysis

We set parameter values to $\delta = \lambda, K = 0.5$ for example. In the case of the parameter conditions of (2), hyperchaos generation is guaranteed if on attractor exists. As shown in Fig. 5, the attractors change with

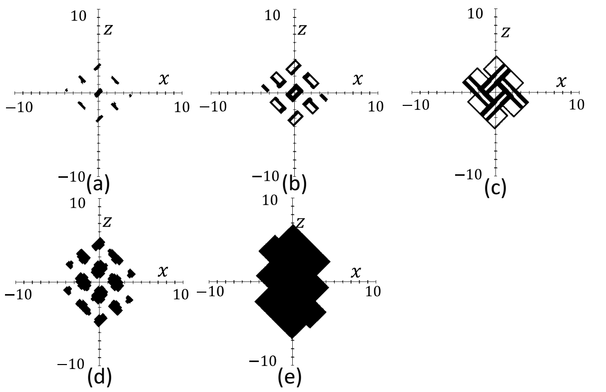


Figure 5: Hyperchaos attractors of 2-D return map ($\delta = \lambda = 0.015, K = 0.5$), (a)$D = 0$, (b)$D = 0.4$, (c)$D = 0.85$, (d)$D = 1.6$, (e)$D = 3.2$

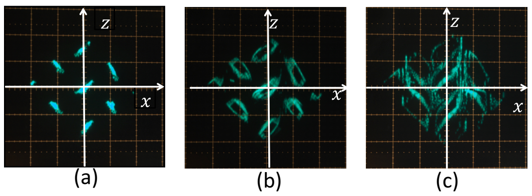


Figure 6: Hyperchaos attractor of return map in laboratory.

changing parameter. Here, we consider the case of D increasing from zero. In Fig. 5(a), the support of positive invariant measure of attractor is split to 9 pieces. We call the case "9 -piece chaos". In Fig. 5(b), the region whose invariant measure is zero appears. This phenomenon is called "Island" [11]. By increasing D further, the number of Island and piece of invariant sets change as Fig. 5(c) and (d), afterward, the attractor is to be one piece chaos without Island as shown in Fig. 5(e). We call such changes bifurcation of hyperchaos. The typical bifurcation can be recognized by experimental circuit, as shown in Fig. 6.

4.1. Shapes of attractor

We consider the edge of attractor. We define the set of points on the line $z = x + D$. The set is threshold for u changing from 1 to -1. In Fig. 7, the line L_{th+} and L_{th-} are the sets of points which are mapped by $\mathbf{F}_+ and \mathbf{F}_-$. L_{th+} and L_{th-} are consistent with part of the edge of the attractor. Here, we define the intersection point (x_{p1}, z_{p1}) of L_{th+} and $z = x+D$ as shown in Fig. 7(a). (x_{p0}, z_{p0}) satisfy the following relationship.

$$x_{p1} = f_+(x_{p0}), z_{p1} = g_+(z_{p0}). \tag{8}$$

Therefore, these points can be obtained exactly as follows.

$$\begin{aligned} x_{p0} &= f_+^{-1}(x_{p1}) = \tfrac{-x_{p1}+(A+1)}{A}, \\ z_{p0} &= g_+^{-1}(z_{p1}) = \tfrac{z_{p1}-(B-1)C}{B}, \end{aligned} \tag{9}$$

$$\begin{aligned} x_{p1} &= \frac{-A(\frac{D-(B-1)C}{B}-D)+A+1}{1+\frac{A}{B}}, \\ z_{p1} &= \frac{B(\frac{D+1}{A}+1+D)+(B-1)C}{1+\frac{B}{A}}. \end{aligned} \tag{10}$$

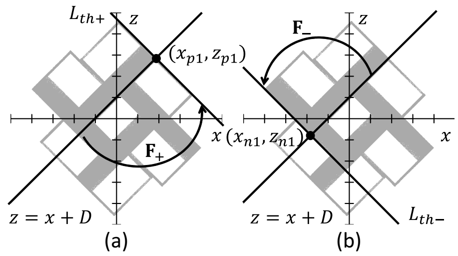


Figure 7: The line of the set mapping the point on the threshold.

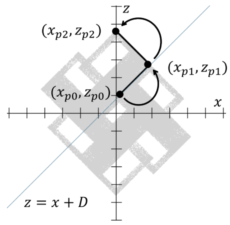


Figure 8: The edge of attractor

In addition, the point (x_{n1}, z_{n1}) as shown in Fig. 7(b) can be determined as follows.

$$\begin{aligned} x_{n1} &= \frac{-\frac{A}{B}D+\frac{A}{B}C-AC+AD-A-1}{1+\frac{A}{B}}, \\ z_{n1} &= \frac{\frac{B}{A}D-B-\frac{B}{A}+BD-BC+C}{1+\frac{B}{A}}, \end{aligned} \tag{11}$$

where $A = e^{\delta\tau}, B = e^{\lambda\tau}$ and $C = \frac{K}{\lambda}$. We assume the segment of a line between the start point (x_{p0}, z_{p0}) and end point (x_{p1}, z_{p1}). We consider the mapping of the set on the segment. The point (x_{p2}, z_{p2}) mapped (x_{p1}, z_{p1}) by f_+ and g_+ is the corner of the attractor as shown in Fig. 8. It is possible to determine all the edges of the attractor by repeating the same procedure. There are two bifurcation phenomena as changing the number of the piece and generation of Islands. We consider the condition of these bifurcation. Bifurcation is related to the unstable periodic point (UPP) in the system and the shapes of attractor. We consider increasing D from 0. In Fig. 9(a) and (b), the both attractors are 9-piece chaos. Islands have occurred in the attractor in Fig. 9(b). Here, we define the UPP $(x_{an}, z_{an}, -1)$ as follows.

$$\begin{aligned} x_{an} &= f_-^{\frac{n}{2}-1}\circ f_+^{\frac{n}{2}}\circ f_-(x_{an}), \\ z_{an} &= g_-^{\frac{n}{2}-1}\circ g_+^{\frac{n}{2}}\circ g_-(z_{an}). \end{aligned} \quad (n = 8, 12, \ldots). \tag{12}$$

The UPPs exist in Islands. Note that the UPPs in Island of Fig. 9(b) correspond to $n = 8$ of (12). Consequently, each islands are referred to as n-island from the corresponding UPP. Islands appear when UPP collides to edge of attractor as shown in Fig. 9. The value of D for bifurcation conditions can be calculated by (11), (12) and (13).

$$z_{an} + x_{an} = z_{n1} + x_{n1}. \tag{13}$$

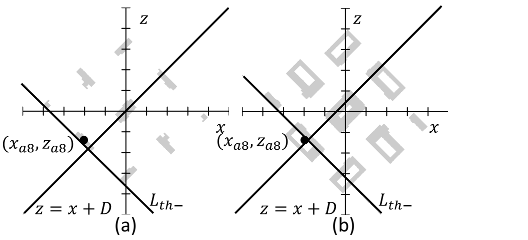


Figure 9: Generation of Islands ((a)$D = 0$, (b)$D = 0.4$).

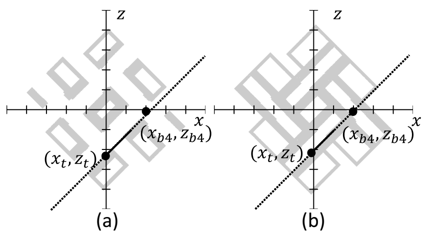


Figure 10: 1-piece and 9-pieces chaos. ((a)$D = 0.5$, (b)$D = 0.7$).

For the other n-Islands, the bifurcation conditions can be obtained. In like manner, it is possible to obtain the condition of dissipating of Island by noting to different edge and UPP. Here, we focus on changing the number of piece of attractor as shown in Fig. 10. This bifurcation also relates to UPP $(x_{bn}, z_{bn}, 1)$ which satisfy.

$$\begin{array}{l} x_{bn} = f_+^{\frac{n}{2}-1} \circ f_-^{\frac{n}{2}} \circ f_+(x_{bn}), \\ z_{bn} = g_+^{\frac{n}{2}-1} \circ g_-^{\frac{n}{2}} \circ g_+(z_{bn}). \end{array} \quad (n = 4, 8, 12, ...). \quad (14)$$

The corner point of the edge related to this bifurcation can describe as follows.

$$x_t = f_-^2 \circ f_+ \circ f_-^2(x_1),\ z_t = g_-^2 \circ g_+ \circ f_-^2(z_1). \quad (15)$$

Bifurcation occurs by the collision of the UPP and the edge. Therefore, the value of D can be calculated by (14), (15) and (16).

$$z_{bn} - x_{bn} = z_t - x_t. \quad (16)$$

For the other other-piece chaos, the bifurcation conditions can be obtained. In like manner, it is possible to obtain the condition of changing the number of piece by noting to different edge and UPP.

4.2. Bifurcation sets

We classify the shape of the attractor and depict the bifurcation sets on parameter plane $(D - \delta)$. The bifurcation sets are shown in Fig. 11(a). In Fig. 11(a), by increasing D, generation of n-Islands is observed on the bifurcation set IS_n, and dissipation of n-Islands is observed on the bifurcation set IE_n. PO_n is bifurcation set from n-piece chaos to 1-piece, PS_n is bifurcation set from 1-piece chaos to n-pieces. In Fig. 11(b), I_n is the existence region of n-Island and P_n is the existence region of n-piece chaos.

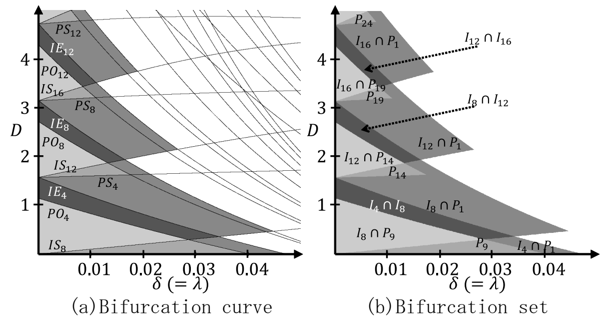


Figure 11: Bifurcation diagram

5. Conclusion

In this paper, we analyzed bifurcation phenomena of 4-D manifold piecewise linear system with hysteresis characteristic by using an embedded return map. We have derived the bifurcation sets and confirmed some bifurcation of hyperchaos in laboratory measurements.

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