# Bifurcation analysis of coupled Izhikevich neuron model with an external periodic force 

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#### Abstract

The neuron model is an important when simulating the behavior of the neurons. There are so many models developed (e.g., Fitz-Hugh-Nagumo, Integrated and Fire model, BVP, etc.) and there are also so many studies about these models[1]-[5]. Especially, the model developed by Izhikevich[1] is focused recently because it can decrease numerical costs at simulation and express all patterns of burstings. In addition, the nonlinear phenomena, e.g. bifurcation phenomena or chaotic attractor, occur at this model since this is nonlinear non-autonomous system. In this paper, we focus on the nonlinear phenomena of coupled Izhikevich neuron model with external periodic force.


## 1. Introduction

In the nonlinear dynamical systems, the bifurcation phenomena are observed by changing parameters. There are some types of local bifurcations just like follows: tangent bifurcation, period-doubling bifurcation and NeimarkSacker bifurcation. They affect qualitative properties of the systems and cause chaotic attractors. When these bifurcations occur, it is important that the bifurcation parameter is clarified analytically. Especially, the non-autonomous nonlinear dynamics are very difficult to analyze. However, non-autonomous dynamics appear at many fields: physics, electrodynamics or biology.

Recently, there are two ways to analyze these dynamic problems in biology. One is an experiment and another is the calculating. For the latter viewpoint, there are so many studies to construct the mathematical model thanks to the technical innovation. From these models, it becomes possible to compare the experimental result and the simulating result. For example, the Hodgkin-Huxley model can simulate the occurrence of neuron's action potential and the integrate-and-fire model can include the past information to the system. Above these models, the Izhikevich neuron model is usually used recently since the model can reduce the calculating costs and express almost all phenomena of neuron's action potential. Moreover, the jump of the state occurs at this model. This jump was, in fact, the reason why the analysis for this model had never been done before, and our previous study showed the solution for this problem.

In our previous study by Tamura[6], it is given that the bifurcation analysis of the Izhikevich neuron model, and the other one by Ito[7] showed the analysis of two-coupled Izhikevich neuron model.

Moreover, our previous study demonstrate availability of the analysis for forced Izhikevich model[8]. On this study, it is proposed that converting the system from nonautonomous to autonomous can solve the bifurcation problem for nonlinear non-autonomous system with jumps. This is the method that had never been developed yet. In this study, we analyze the coupled Izhikevich neuron model with external force, which is 4-dimensional nonlinear nonautonomous system with the jumps.

## 2. Analysis Method

First of all, the Izhikevich model is described as

$$
\begin{array}{r}
\frac{d \boldsymbol{v}}{d t}=\boldsymbol{f}(\boldsymbol{v}, \lambda, I)=\binom{0.04 v^{2}+5 v+140-u+I}{a(b v-u)}, \\
\text { if } v \geq 30[\mathrm{mV}], \text { then }\left\{\begin{array}{l}
v \leftarrow c \\
u \leftarrow u+d
\end{array},\right.  \tag{2}\\
\boldsymbol{v}=(v, u)^{\top} \in \boldsymbol{R}^{2}, \lambda=(a, b, c, d)^{\top} \in \boldsymbol{R}^{4}
\end{array}
$$

where, $v$ is membrane potential of the neuron, $u$ is variable for restoration, $I$ is external force, $a$ is time scale of $u, b$ is sensibility of $u$ for $v, c$ is voltage after the bursting and $d$ is strength of restoration. The $a, b, c$, and $d$ are static parameters and they have already derived experimentally as shown in Table1. The time wave following this system is just like below Fig.1. To express the coupling of 2 neurons, the

Table 1: Parameters of the Izhikevich model

| time scale of $u(a)$ | 0.2 |
| :--- | :---: |
| sensibility of $u$ for $v(b)$ | 0.2 |
| voltage after the bursting $(c)$ | -50 |
| strength of restoration $(d)$ | 2 |

gap junction is well known. In addition, since the parameter $I$ is an external force, the periodic force can be added to the system so simply. Thus, the system we analyze is


Figure 1: Sample wave form of $v$.
described as following:

$$
\left\{\begin{align*}
\frac{d \boldsymbol{v}_{\alpha}}{d t} & =\boldsymbol{f}\left(\boldsymbol{v}_{\alpha}, \lambda, I_{0}+I \cos \omega t\right)+\binom{\delta\left(v_{\beta}-v_{\alpha}\right)}{0} \\
& =\left(\begin{array}{l}
0.04 v_{\alpha}^{2}+5 v_{\alpha}+140-u_{\alpha}+I_{0}+I \cos \omega t \\
a\left(b v_{\alpha}-u_{\alpha}\right) \\
+\delta\left(v_{\beta}-v_{\alpha}\right)
\end{array}\right) \\
\frac{d \boldsymbol{v}_{\beta}}{d t} & =\boldsymbol{f}\left(v_{\beta}, \lambda, I_{0}\right)+\binom{\delta\left(v_{\alpha}-v_{\beta}\right)}{0} \\
& =\binom{0.04 v_{\beta}^{2}+5 v_{\beta}+140-u_{\beta}+I_{0}+\delta\left(v_{\beta}-v_{\alpha}\right)}{a\left(b v_{\beta}-u_{\beta}\right)} \tag{3}
\end{align*}\right.
$$

where, $\boldsymbol{v}_{i}=\left(v_{i}, u_{i}\right)$ are each neuron's states, $\delta$ is a coupling coefficient, $I$ is an amplitude and $\omega$ is frequency of the external force. Since the model is a nonlinear nonautonomous system with jumps, it is necessary to use the special method such as our previous study[8]. Following this[8], Eq.(3) is converted to next:

$$
\begin{align*}
\frac{d z}{d t}= & \boldsymbol{g}\left(z, \lambda, \lambda_{\mathrm{in}}\right) \\
= & \left(\begin{array}{l}
\boldsymbol{f}\left(\boldsymbol{v}_{\alpha}, \lambda, I_{0}+I \cos \omega z\right)+\left(\begin{array}{l}
\delta\left(v_{\beta}-v_{\alpha}\right) \\
0 \\
\boldsymbol{f}\left(\boldsymbol{v}_{\beta}, \lambda, I_{0}\right)+\left(\begin{array}{l}
\delta\left(v_{\alpha}-v_{\beta}\right) \\
0 \\
1
\end{array}\right)
\end{array}\right) \\
\\
z=\left(\begin{array}{c}
\boldsymbol{v}_{\alpha} \\
\boldsymbol{v}_{\beta} \\
z
\end{array}\right), \lambda_{\mathrm{in}}=\left(I_{0}, I, \omega, \delta\right)^{\top} .
\end{array},\right.
\end{align*}
$$

This is a converting from non-autonomous to autonomous.
Secondary, it is necessary to define some functions, mappings and so on. For example, the solution orbit is defined
as follows:

$$
\left(\begin{array}{c}
v_{\alpha}(t)  \tag{5}\\
u_{\alpha}(t) \\
v_{\beta}(t) \\
u_{\beta}(t) \\
z(t)
\end{array}\right)=\varphi=\left(\begin{array}{c}
\varphi_{1}\left(t, z_{0}, \lambda\right) \\
\varphi_{2}\left(t, z_{0}, \lambda\right) \\
\varphi_{3}\left(t, z_{0}, \lambda\right) \\
\varphi_{4}\left(t, z_{0}, \lambda\right) \\
\varphi_{5}\left(t, z_{0}, \lambda\right)
\end{array}\right)
$$

where, $z_{0}=\left(v_{\alpha 0}, u_{\alpha 0}, v_{\beta 0}, u_{\beta 0}, z_{0}\right)$ are the initial values of state variables. In addition, it is important to consider about jump (in other word; mapping) just as following composition:

$$
\begin{equation*}
T=T_{m} \circ T_{m-1} \circ \cdots \circ T_{1} \circ T_{0}, \tag{6}
\end{equation*}
$$

where, $T_{k}: z_{k} \rightarrow z_{k+1}$ are the mappings between sections defined by Eq.(2) and $z_{k}$ is the states when the orbit of the system reaches the section. The concept image of jump is below Fig.2. On the Fig.2, there are 2 sections that


Figure 2: Concept figure of Jump.
expressed as Action potential and Poincaré section. Actions potential section is defined as $\Pi_{1}=\left\{z \in \boldsymbol{R}^{5} ; q_{1}(z)=\right.$ $\left.v_{\alpha}-30=0\right\}, \Pi_{2}=\left\{z \in \boldsymbol{R}^{5} ; q_{1}(z)=v_{\beta}-30=0\right\}$. On the other hand, when the mapping $T$ follows the rule below:

$$
\begin{equation*}
T\left(z_{0}\right)-z_{0}=0 \tag{7}
\end{equation*}
$$

the $T$ is called Poincaré mapping. Since the system is non-autonomous, the period of Poincaré mapping becomes $2 \pi / \omega$. Thus, the Poincaré section is defined as $\Pi_{3}=\{z \in$ $\left.\mathbf{5}^{3}, \omega \in \boldsymbol{R} ; q_{2}(z, \omega)=\sin (\omega z / 2)=0\right\}$.

At last, let us introduce the method to solve the bifurcation problem. To solve the bifurcation problem, it is necessary to obtain the Jacobian matrix of the Poincaré mapping. From Eq. 6 and chain rule of derivatives, the Jacobian matrix $J$ is derived as follows:

$$
\begin{align*}
J= & \left.\left.\frac{\partial T_{k}}{\partial z_{k}}\right|_{t=\tau_{k}} \circ \frac{\partial T_{k-1}}{\partial z_{k-1}}\right|_{t=\tau_{k-1}} \\
& \left.\left.\ldots \ldots \circ \frac{\partial T_{1}}{\partial z_{1}}\right|_{t=\tau_{1}} \circ \frac{\partial T_{0}}{\partial z_{0}}\right|_{t=\tau_{0}} \\
= & \left.\prod_{k=m-1}^{0} \frac{\partial T_{k}}{\partial z_{k}}\right|_{t=\tau_{k}}, \tau=\sum_{i=0}^{m-1} \tau_{i}=2 n \pi / \omega, \tag{8}
\end{align*}
$$

where, $\tau_{k}$ is the time when the orbit of the system reaches the section. When computing the Jacobian matrix of the system with the jump, it is necessary to apply the information of the action potential section and the Poincaré section to the Jacobian matrix because the time when the orbit reaches to the section depends on the initial values and each location of the section, thus,

$$
\begin{equation*}
\left.\frac{\partial T_{k}}{\partial z_{k}}\right|_{t=\tau_{k}}=\left.\left[I_{n}-\left.\frac{1}{\left.\frac{\partial q_{i}}{\partial z} \boldsymbol{g}\right|_{t=\tau_{k}}} \boldsymbol{g}\right|_{t=\tau_{k}} \frac{\partial q_{i}}{\partial z}\right] \frac{\partial \boldsymbol{\varphi}}{\partial z_{k}}\right|_{t=\tau_{k}} \tag{9}
\end{equation*}
$$

By the way, it is a generally accepted theory that the bifurcation phenomena occur when the $\mu$ follows the rule: $|\mu|=1$, where, $\mu$ is characteristic multiplier of $J$. For example, if $\mu=-1$, the period-doubling bifurcation occurs and when $\mu=1$, the tangent bifurcation occurs. Even if $\mu$ is a complex number, the bifurcation occurs if the $|\mu|=1$. This bifurcation is called Neimark-Sacker bifurcation. If the $J$ is obtained exactly, these $\mu$ are obtained by solving next:

$$
\begin{equation*}
\chi(\mu)=\operatorname{det}(J-\mu I)=0 \tag{10}
\end{equation*}
$$

Since the $J$ has been obtained exactly by the method previously explained, the $\mu$ can be obtained accurately. At this time, the $J$ depends on the initial values $z_{0}$ and the parameters $\lambda_{i n}$. Conversely, when getting the parameter at bifurcation point, the value of $\mu$ at Eq.(10) must be set as $|\mu|=1$. By setting like this, Eq.(10) becomes the equation only for $z_{0}$ and $\lambda_{i n}$. Since the equations for $z_{0}$ have been proposed at Eq.(7), the parameter of bifurcation point is obtained by solving the simultaneous equations:

$$
\left\{\begin{array}{ll}
T\left(z_{0}\right)-z_{0} & =0  \tag{11}\\
\chi(\mu) & =0 \\
|\mu|-1 & =0
\end{array} .\right.
$$

In this study, the Newton's method is effective to solve Eq.(11).

## 3. Result

The bifurcation curves have been obtained on the Fig.3. In this figure, $G_{j}^{i}$ means tangent bifurcation of $i$-periodic solution, $I_{j}^{i}$ means period-doubling bifurcation of $i$-periodic solution and $N S_{j}^{i}$ means Neimark-Sacker bifurcation of $i$ periodic solution. At the point $A$, the solution orbit is quasiperiodic orbit(Fig.4). At the point $B$, the solution orbit becomes 1-periodic orbit by crossing $G_{1}^{1}$ (Fig.5). At the point $C$, the solution orbit is still 1-periodic orbit(Fig.6). At the point $D$, the solution orbit becomes 2-periodic orbit by crossing $I_{3}^{1}$ (Fig.7). Through the all, quasi-periodic orbit appears by crossing tangent bifurcation's curve. Moreover, stable fields of this system are not so wide because there are so many fields of quasi-periodic orbits.


Figure 4: Phase portrait with $I_{0}=10, \delta=2, I=4, \omega=1$


Figure 5: Phase portrait with $I_{0}=10, \delta=2, I=5, \omega=1$


Figure 6: Phase portrait with $I_{0}=10, \delta=2, I=6, \omega=1$


Figure 7: Phase portrait with $I_{0}=10, \delta=2, I=7, \omega=1$


Figure 3: Bifurcation curves with $I_{0}=10, \omega=1$.

## 4. Conclusion

So far, we can not analyze the non-autonomous nonlinear system with jump because of a problem. However, the problem has been solved by our previous study[8]. And now, we show the result that our method, which can solve the non-autonomous nonlinear system with jump can apply to high-dimensional system.

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