A systematic approach to find periodic sinks of the Hénon map close to the classical case

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Abstract— Using the combination of a method to find low period unstable periodic orbits and the continuation method it is demonstrated that there exist low-period sinks extremely close to the classical parameter values of the Hénon map. The problem why these low-period sinks are rarely observed in computer simulations is discussed.

1. Introduction

The Hénon map [1] is a two-parameter map of the plane defined by $h(x, y) = (1 + y - ax^2, bx)$. In spite of extensive study, the long term dynamics of the Hénon map for the classical parameter values $(\bar{a}, \bar{b}) = (1.4, 0.3)$ remains unknown.

When $b = 0$, the Hénon map reduces to the quadratic map $f(x) = 1 - ax^2$. The set of parameter values for which the dynamics of f is regular (the unique attracting set is a periodic sink) is open and dense. On the other hand, the set of parameter values with chaotic dynamics is a Cantor set with positive Lebesgue measure [2]. When $b > 0$ is sufficiently small the set of parameter values a with chaotic dynamics is also a Cantor set with positive Lebesgue measure [3].

However, little is known for parameter values close to the classical case. In [4], results of search for parameter values in the region $(a, b) \in (0, 2) \times (0, 0.5)$ for which there exist at least three attractors are reported; several such regions are found. In [5], results of a brute force numerical search for points in parameter space close to $(\bar{a}, \bar{b}) = (1.4, 0.3)$ for which there exists a sink is presented. The search method is based on monitoring trajectories and looking for periodic steady state behaviour. A number of points in parameter space supporting a sink are located; for example it is shown that for $(a, b) = (1.4, 0.2999999774905)$ there exists a period-28 sink. It is shown that close to the classical case, the regions of existence of sinks are very narrow, the sinks have very small immediate basin size, and finding them is not a trivial numerical task. In spite of very long computation times, only a limited number of sink regions have been found, and the hypothesis that the number of sink regions with a given period p increases with p has not been confirmed.

In this work, we continue the study of the long term behavior of the Hénon map for (a, b) close to the classical case. We propose a systematic method to search for low-period sinks in a specified region, and report results of applying this method to search for low-period sinks in the region $Q_0 = [1.3999, 1.4001] \times [0.2999, 0.3001]$ of the parameter space. The method is based on finding all periodic orbits existing for fixed parameter values using the Biham-Wenzel method [6], and then using the continuation method in the parameter space to find a sink. This approach allows us to find many more sink existence regions and in consequence find sinks for parameter values much closer to the classical case than using the monitoring trajectory based method [5]. We discuss the problem why in spite of very long observation times it is difficult to find these sinks in simulations.

2. A systematic method to find sinks

We say that z_0 is a period-p point if $z_0 = h^p(z_0)$ and $z_0 \neq h^k(z_0)$ for $0 < k < p$. We say that z_0 or the orbit $(z_0, z_1, \ldots, z_{p-1})$ is a *period-p sink* if z_0 is a period-p point and the trajectory (z_k) is asymptotically stable, i.e., for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|z - z_l\| < \delta$ for some $l = 0, 1, \ldots, p-1$ then $\|h^k(z) - h^k(z_l)\| < \epsilon$ for all $k > 0$ and $\lim_{k\to\infty} \|h^k(z) - h^k(z_l)\| = 0$.

In this section we present a systematic method to locate low-period sinks in a specified region of the parameter space. The method is composed of two steps. In the first step, for selected points in the parameter space, locations of low-period orbits in the state space are found. Usually, none of these periodic orbits is stable. In the second step, for each unstable periodic orbit found, we continue the solution, via a move in the parameter space, towards a sink.

2.1. Detecting periodic orbits

Let us briefly present the Biham-Wenzel method [6] to locate all low-period unstable cycles of the Hénon map. This method is selected because of its speed, and capability of locating correctly positions of all periodic orbits for relatively large periods. Moreover, for this method some improvements, which will be described below, are possible, which significantly reduce time necessary to find all periodic orbits of a given period. Another choice is to use a general purpose rigorous method to find all low-period cycles [7]. In this method an interval operator is combined with the generalized bisection technique to perform an exhaustive search of the state space for periodic orbits of a given length. A rigorous method is used here to validate the results obtained for periods $p \leq 30$.

The Biham-Wenzel method to find period-p cycles is based on the construction of artificial continuous dynamical systems of order p defined by

$$\frac{\mathrm{d}w_k}{\mathrm{d}t} = s_k\left(-w_{k+1} + a - w_k^2 + bw_{k-1}\right), \quad 0 \leq k < p \qquad (1)$$

where $(w_0, w_1, \ldots, w_{p-1})$ is the state vector, $w_{-1} := w_{p-1}$, $w_p := w_0$, and $s = (s_0, s_1, \ldots, s_{p-1})$ is a symbol sequence, $s_k \in \{-1, +1\}$. Note that if $(z_0, z_1, \ldots, z_{p-1})$ is a periodic orbit with $z_k = (x_k, y_k)$ then $(w_0, w_1, \ldots w_{p-1})$ defined by $w_k = ax_k$ satisfies $-w_{k+1} + a - w_k^2 + bw_{k-1} = 0$ for all $k = 0, 1, \ldots p-1$. This can be seen by noting that $w_{k+1} = ax_{k+1} = a(1 - ax_k^2 + bx_{k-1}) = a - w_k^2 + bw_{k-1}$. It follows that there is a one-to-one correspondence between fixed points of h^p and equilibria of (1). In [6], it is claimed that for each fixed point of h^p there is exactly one symbol sequence s for which the corresponding equilibrium of (1) is stable. Hence, in order to find all fixed points of h^p, it is proposed to find steady state behaviors for all possible symbol sequences s of length p. The system (1) is integrated until either the right hand side of (1) becomes sufficiently small (it is proposed to use the value $\varepsilon = 10^{-7}$), which means that periodic orbit has been found, or the norm of the solution w_k becomes sufficiently large, which indicates that the solution escapes to infinity. If an equilibrium is stable, a trajectory converges to it for initial conditions which are small with respect to $\sqrt{a}$ (in the following we use initial conditions $w_k = 0$). Since the interest is in the steady state only, one can use a simple integration method with a relatively large time step. We use the fourth-order Runge-Kutta method with the step size 0.1. Eliminating cyclic permutations and sequences for which the primary period is not p reduces the number of sequences to be considered by at least a factor of p, for example when $p = 33$ the number of sequences to be considered is $260300986 \approx 2.6 \cdot 10^8$, while the total number of sequences of length 33 is $2^{33} \approx 8.6 \cdot 10^9$.

In Table 1, we report results obtained for $p \leq 33$ for the classical parameter values using the method described above. Results for $p \leq 28$ have already been presented in [6]. We show the number P_p of period-p orbits, the number Q_p of fixed points of h^p, and the estimate $\mathrm{H}_p = p^{-1} \log \mathrm{Q}_p$ of the topological entropy of the Hénon map based on the number of fixed points of h^p. The results shown in Table 1 agree with the rigorous results for $p \leq 30$ presented in [7], which means that the Biham-Wenzel method works properly for relatively large periods.

Table 1: The number of periodic orbits for the Hénon map, $a = 1.4$, $b = 0.3$ found using the Biham-Wenzel method

p	P_p	Q_p	H_p
18	233	4264	0.4643313
19	364	6918	0.4653622
20	535	10808	0.4644021
21	834	17544	0.4653556
22	1225	27108	0.4639811
23	1930	44392	0.4652528
24	2902	69952	0.4648152
25	4498	112452	0.4652113
26	6806	177376	0.4648472
27	10518	284042	0.4650695
28	16031	449520	0.4648548
29	24740	717462	0.4649474
30	37936	1139276	0.4648635
31	58656	1818338	0.4649495
32	90343	2892672	0.4649279
33	139674	4609398	0.4649578
34	215597	7333124	0.4649386
35	333558	11674560	0.4649406
36	516064	18582800	0.4649374
37	799372	29576766	0.4649324
38	1238950	47087020	0.4649344
39	1921864	74953114	0.4649326
40	2983342	119344544	0.4649381
41	4633278	189964400	0.4649353

Further savings in computation time for longer periodic orbits can be achieved by skipping sequences containing forbidden subsequences (compare also the idea of pruning [8]). It has been found that for $(a, b) = (1.4, 0.3)$ admissible sequences of length $p \leq 33$ apart from the fixed point with symbol sequence $s = (-)$ located outside the numerically observed attractor do not contain subsequences $(----)$, $(-++-)$, and $(--+-)$, i.e. these three subsequences are forbidden. This property has also been confirmed for the four corners of $Q_0 = [1.3999, 1.4001] \times [0.2999, 0.3001]$. This indicates that we can exclude periodic sequences containing these three subsequences when searching for periodic orbits within Q_0. Excluding forbidden subsequences reduces the number of sequences to be considered. For example, for $p = 33$ the number of sequences is reduced from 260300986 to 902317. Even further savings in computation time can be achieved by skipping longer forbidden subsequences. We have verified that 61977 out of $2^{16} = 65536$ subsequences of length 16 do not appear in any admissible sequence of period $p \leq 33$. As before, the sequence $(-)$ corresponding to one of the fixed points was excluded. The set of 61977 forbidden subsequences of length 16 can be simplified to 28 subsequences of various length not larger than 16. Skipping these forbidden subsequences when searching for period-33 orbits decreases the number of sequences to be considered to 259390. The results obtained for the classical parameter values and periods $p \leq 41$ using the procedure presented above are reported in Table 1. Let us note that the estimate $\mathrm{H}_p = p^{-1} \log \mathrm{Q}_p$ of the topological entropy stabilizes around 0.46493. The five most significant digits are constant for $p \geq 36$. Hence, this number can be considered as a good approximation of the topological entropy of the Hénon map with classical parameter values. This is a by-product of our search procedure.

Similar computations have been performed for points in the parameter space being corners of the square Q_0. These results will be used in the following section to find regions of existence of sinks having non-empty intersection with the square Q_0.

We would like to stress that although the results obtained using the Biham-Wenzel method agree with the true results for periods $p \leq 30$ one cannot treat this method as a rigorous one. We have observed that the number of periodic orbits found depends on the parameters of the method. For example, the period-33 sequence $s = (---+++-+-+-+--+++-+-+-+++--++++-+)$ was found admissible for $a = 1.4001$, $b = 0.3001$ when $\varepsilon = 10^{-7}$ was used and non-admissible for $\varepsilon = 10^{-8}$. The reason for this behaviour is the fact that the region of existence of the periodic orbit with the symbol sequence s starts very close to the point $(1.4001, 0.3001)$.

2.2. The continuation method to find sinks

Low-period cycles existing for certain points in the parameter space will now be used to locate sink regions intersecting the square $Q_0 = [1.3999, 1.4001] \times [0.2999, 0.3001]$.

There are two possible approaches to achieve this goal. The first one is to continue all periodic orbits found in the previous step. Continuation is carried out in a direction in which the spectral radius of the Jacobian matrix computed along the periodic orbit decreases. When at a certain point obtained during the continuation procedure, the spectral radius of the Jacobian matrix is smaller than 1 the procedure is stopped—the sink region has been found. If the continuation procedure leads outside Q_0 it is also stopped with no sink. Since the number of period-p orbits grows very fast with p this approach becomes infeasible for larger p.

In the second approach, we select the following five points A_k: four corners of Q_0 and its center. Next, we find symbol sequences which are admissible at some of points A_k but not all. Such sequences are called *missing sequences*. Finally, for each missing sequence we continue from points where this sequence is admissible to find the corresponding sink region. To confirm that the sink found is not a rounding error artifact, we prove its existence using the interval Newton method (for details see [5]).

We now explain why this approach allows us to find most sink existence regions intersecting Q_0. Let us first recall how periodic orbits emerge in the parameters space. In the fold bifurcation a pair of periodic orbits is born. For the Hénon map, one of them is stable and the other is unstable. In the period-doubling bifurcation a periodic orbit looses its stability and a stable orbit with twice the period is born. Hence, one can expect that at the border of the periodic orbit existence region the cycle may be stable. Therefore, in order to find sink regions we need to detect borders of periodic orbit existence regions. Usually, if the border of existence region intersects Q_0, the corresponding symbol sequence is admissible for some but not all points A_k. This happens in most situations because locally existence regions close to the border look like a halfplane, and Q_0 is small, and hence intersections of borders and Q_0 are usually either empty or are (almost) straight intervals. Situations when the border of an existence regions turns inside Q_0 (three such examples are given in [5]) are very rare, which means that the proposed method detects most borders of periodic orbit existence regions intersecting Q_0.

Table 2: The number of missing sequences M_p and the number R_p of sink regions in Q_0 with period p

p	M_p	R_p	p	M_p	R_p
18	2	1	30	412	206
19	4	2	31	624	312
20	6	3	32	1014	507
21	4	2	33	1654	826
22	10	5	34	2580	1290
23	10	5	35	4192	2096
24	24	12	36	6545	3272
25	34	17	37	10514	5250
26	60	30	38	16578	—
27	110	55	39	26599	—
28	160	80	40	42112	—
29	248	124	41	67523	—

The number of missing sequences of length p found for Q_0 versus p is shown in Table 2. Since the number of missing sequences is significantly smaller than the number of admissible sequences (compare Table 1), the second approach is much faster than the first one.

For all missing sequences with period $p \leq 37$, corresponding periodic orbits have been continued to find sink existence regions. The results are shown in Table 2. If all periodic orbits are created via fold bifurcations, we expect that half of the missing sequences correspond to sinks (the other half corresponds to unstable periodic orbits). Note that the number R_p of sink regions found is half the number of missing sequences M_p for $p \leq 35$ with the exception of $p = 33$, for which R_p is one less than half of M_p. This indicates that indeed there are stable/unstable orbits pairs in the set of missing sequences. The discrepancy for $p = 33$ has been explained at the end of the previous section. For $p \geq 36$ there are more regions passing close to the borders of Q_0 which may cause a failure of the Biham-Wenzel method and in consequence a violation of the condition $\mathrm{M}_p = 2\,\mathrm{R}_p$. The total number of detected sinks regions with period $p \leq 37$ is 14095, which is much more than 461 sink regions found using the monitoring trajectory approach (compare [5]). The number of missing sequences and the number of sink regions grow exponentially with the period which numerically confirms the hypotheses that sink regions densely fill the parameter space.

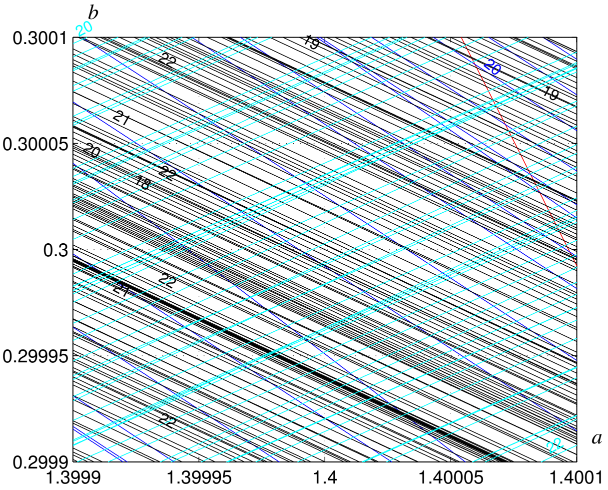

Figure 1: Existence regions of sinks with period $p \leq 28$

212 sink existence regions with period $p \leq 28$ are plotted in Fig. 1. Sink existence regions have been found using a version of the continuation method designed for narrow regions described in detail in [5]. Sink regions with period $p \leq 22$ are labelled.

We have carried out similar computations for $p \geq 38$ and a much smaller square centered at $(1.4, 0.3)$ with the goal of finding a sink as close to the classical values as possible. Decreasing the size of the square reduces the number of missing sequences, and thus shortens the computation time. The closest sink we are able to find is the period-41 sink existing for $(a, b) = (1.3999999997706479, 0.29999999958655875)$. The distance between this point and $(1.4, 0.3)$ is less than $4.73 \cdot 10^{-10}$, which is more than 40 times closer than the closest period-28 sink reported in [5]. The width of the existence region is approximately $3.04 \cdot 10^{-17}$. The minimum immediate basin radius of the sink is $\mathrm{r}_\varepsilon = 3.31 \cdot 10^{-17}$, which means that if we perturb the periodic solution by more than $3.31 \cdot 10^{-17}$, we cannot be sure that the trajectory converges to the sink (for a precise definition of r_ε see [5]). In order to detect this sink we have to carry out the computations in a higher precision than r_ε. We have verified that this sink appears to be unstable when the computations are performed in standard double precision; a trajectory escapes from the sink even if is it started exactly (with the double precision) at the sink position. Using multiple precision GPU software [9], we have estimated that on average the number of iterations needed to converge to the sink is $4.4 \cdot 10^{12}$. This means that even if the computations are done with sufficient precision, we need a large number of iterations to observe this sink starting from random initial conditions. This explains why this sink region and many other low period sink regions reported in Table 2 were not found in simulations in spite of very long computation times (compare [5]).

3. Conclusion

The results obtained provide numerical support for the belief that the set of parameter values with a periodic sink is dense in a neighborhood of $(1.4, 0.3)$. By analogy with the quadratic map, it is also expected that in a neighborhood of $(1.4, 0.3)$ the set of parameter values with chaotic behaviour is a Cantor set with positive Lebesgue measure. It has been confirmed that sink existence regions are very narrow, and that the transient times to converge to a sink can be extremely long.

Acknowledgments

This work was supported in part by the AGH University of Science and Technology, grant no. 11.11.120.343, and by the Swedish Research Council, grant no. 2007-523.

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