Intermittency: uniqueness of the period-3 in the logistic map

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Abstract—In this communication the period-3 intermittency in the logistic map is further investigated. Even if in the literature the period-3 parameter a is often reported, it does not appear a general analytical proof to establish its uniqueness. A new method it is proposed to this aim.

1. Introduction

The logistic map [1] is the most studied chaotic discrete-time system. Its simplicity marks all the features of complex dynamical behavior. Route to chaos, intermittency, bifurcations, can be easily and analytically investigated in the logistic map due to its very easy representation. The published papers on the logistic map are continuously increasing since 1978 when the map has been presented by the biologist Robert May [2]. The literature adds day to day new results regarding the logistic map. Recently, paper regarding also the period-3 behavior and intermittency have been proposed [3]. Analytical solutions to the problem of locating the period-3 intermittency have presented before [4, 5, 6]. However, in these papers the analytical derivation of the value of a for which a period-3 motion is obtained can not ensure the uniqueness of the $a = 1 + \sqrt{8}$ value.

In this paper, a new analytical method to prove that intermittency with period-3 occurs for one only value of the parameter a is proposed. Odd intermittency of period-5, for example, can appear for different values of a in the logistic map. The presented proof confirms the uniqueness of the period-3 intermittency condition.

The paper is organized as follows: in Section 2 some details on the intermittency have been reported, in Section 3 the proof of the uniqueness of period-3 in the logistic map is presented. Finally, in Section 4 the conclusive remarks will be described.

2. Intermittency

Briefly, intermittency behavior shows recurrent long phases of almost resting behavior corresponding to almost perfect cycle oscillations, called *laminar phase* alternated with unpredictable jump in chaotic behavior. The continuous burst between chaos and period behavior is the essence of the intermittency. Intermittency is also a condition for transient chaos [7].

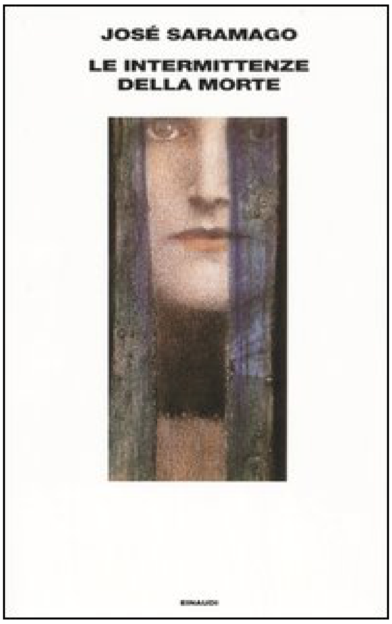

Figure 1: Italian edition of *Death with interruptions* by José Saramago.

Reading the book by José Saramago *Death with interruptions* [8, 9], Fig. 1, one can deeply perceive the meaning of intermittency.

In the case of intermittency related to period-3 in the logistic map, the behavior can be observed in the numerical experiments shown in Fig. 2. Differently from chaotic transient, in which chaotic oscillations relax into a periodic orbit, intermittency consists in the temporal sequence of period-3 oscillations erratically alternated with chaotic windows, like those reported in Fig. 2. The parameter value for which the period-3 behavior does exist is reported in the literature for the quadratic iterator $y(k+1) = ay(k)(1 - y(k))$ to be $a = 1 + \sqrt{8}$. We propose in this paper a new method that further prove the correctness of the parameter and establish its uniqueness.

3. Analytical proof of the uniqueness of the period-3 parameter in the logistic map

Let us iterate the logistic map for searching a period-3 cycle:

$$x_1 = ax_3(1 - x_3) \tag{1}$$

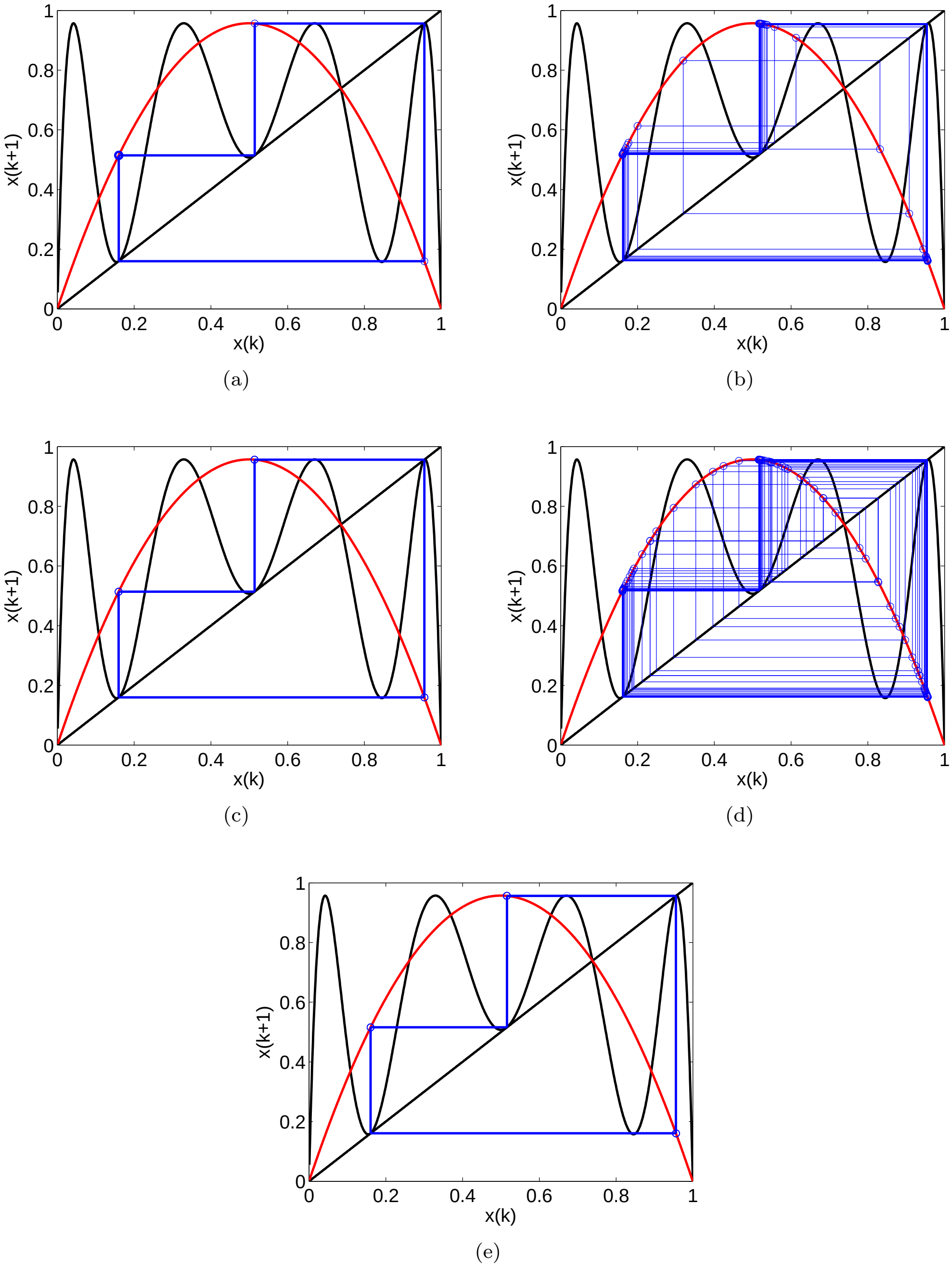

Figure 2: Intermittency around period-3 cycle in the logistic map obtained for $a = 3.8284$: (a) period-3 oscillations for $k \in [200, 350]$, (b) chaotic window for $k \in [350, 500]$, (c) period-3 oscillations for $k \in [500, 600]$, (d) a new chaotic window for $k \in [600, 700]$, and again a period-3 oscillation for $k \in [800, 850]$.

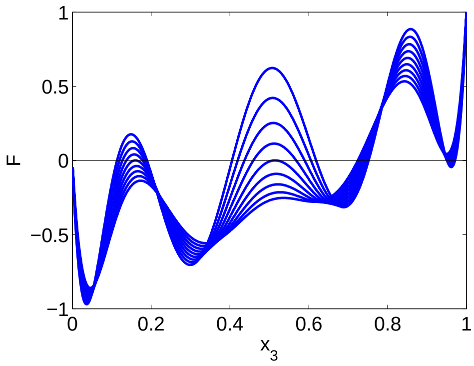

Figure 3: Different curves in the family F for different values of a.

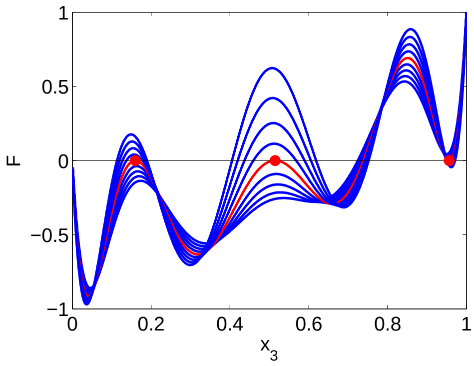

Figure 4: Condition to get a period-3 cycle: the red curve is obtained for $a = 1 + \sqrt{8}$, the three points of the cycle are marked in red.

$$x_2 = ax_1(1 - x_1) \tag{2}$$

$$x_3 = ax_2(1 - x_2) \tag{3}$$

where x_1, x_2, and x_3 are the periodic points of the map and $a \in [0, 1]$. Substituing Eq. (1) in Eq. (2) and in Eq. (3) the following equation can be derived:

$$\begin{aligned} x_3 = &-a^3x_3(a^2x_3(ax_3(x_3-1)+1)(x_3-1)+1)\cdot \\ &\cdot(ax_3(x_3-1)+1)(x_3-1) \end{aligned} \tag{4}$$

Now we define the polynomial F as:

$$\begin{aligned} F = &\, x_3 + a^3x_3(a^2x_3(ax_3(x_3-1)+1)(x_3-1)+1)\cdot \\ &\cdot(ax_3(x_3-1)+1)(x_3-1) \end{aligned} \tag{5}$$

which identifies a family of curves depending on a as reported in Fig. 3.

The condition that allows us to obtain the period-3 is shown in Fig. 4 where the three points of the period-3 cycle are marked. This condition consists in the fact that the iterator can assume only three values, namely the values corresponding to the markers in Fig. 4 i.e. the tangent points of F with the $F = 0$ axis. In order to derive the analytical condition to obtain a, function F has to be divided for the quantity $x_3(x_3 - \frac{a-1}{a})$ since $x_3 = 0$ and $x_3 = \frac{a-1}{a}$ are the two fixed points of the quadratic iterator. The quotient q of the previous polynomial division can be calculated as:

$$\begin{aligned} q = &\, a^7x_3^6 + \\ &+(-3a^7 - a^6)x_3^5 + \\ &+(3a^7 + 4a^6 + a^5)x_3^4 + \\ &+(-a^7 - 5a^6 - 3a^5 - a^4)x_3^3 + \\ &+(2a^6 + 3a^5 + 3a^4 + a^3)x_3^2 + \\ &+(-a^5 - 2a^4 - 2a^3 - a^2)x_3 + \\ &+a^3 + a^2 + a \end{aligned} \tag{6}$$

The existence of a period-3 can be found considering that the parametric polynomial F must divisible without remainder by the polynomial $F_1 = (x_3^3 + Ax_3^2 + Bx_3 + C)^2$, ensuring that solutions are three with double multiplicity. This is necessary since the solutions of F are the possible values of the logistic map for the given parameter value: since the two trivial solutions corresponding to fixed points have been already removed, only three solutions can be obtained. Therefore, the parametric remainder R of the polynomial division $\frac{F}{F_1}$ must be zero:

$$R = c_1x_3^5 + c_2x_3^4 + c_3x_3^3 + c_4x_3^2 + c_5x_3 + c_6 = 0 \tag{7}$$

where the coefficient of the polynomial R are:

$$\begin{aligned} c_1 &= -2Aa^7 - a^6 - 3a^7 \\ c_2 &= a^5 + 4a^6 + 3a^7 - a^7(A^2 + 2B) \\ c_3 &= -a^7(2C + 2AB) - a^4 - 3a^5 - 5a^6 - a^7 \\ c_4 &= (-B^2 - 2AC)a^7 + 2a^6 + 3a^5 + 3a^4 + a^3 \\ c_5 &= -2BCa^7 - a^5 - 2a^4 - 2a^3 - a^2 \\ c_6 &= -C^2a^7 + a^3 + a^2 + a \end{aligned} \tag{8}$$

Posing $c_1 = 0$, we derive $A = -\frac{3a+1}{2a}$. Substituting the calculated value in $c_2 = 0$, the second coefficient can be obtained as $B = \frac{3a^2+10a+3}{8a^2}$. The third coefficient is then obtained substituting A and B in $c_3 = 0$ deriving $C = \frac{a^3-7a^2-5a-5}{16a^3}$. The three remaining polynomials, i.e. c_4, c_5, and c_6, are only in a, hence, if there exist common solutions inside the allowed range $[0, 4]$, a period-3 motion can be observed for those values of a. Among the possible solutions, there are only two of them which are common to the three polynomials, i.e. $a = 1 + \sqrt{8}$ and $a = 1 - \sqrt{8}$. But the latter solution must be discarded since it falls outside the allowed range of values for a, and thus there is only one root, $a = 1 + \sqrt{8}$, which leads to a period-3 trajectory.

This further confirms the correctness of the value of a necessary to have a period-3 cycle and establish its uniqueness.

4. Conclusion

The uniqueness of a period-3 cycle for the logistic map has been analytically proved. This is not true in general for odd periodic cycles. For example, period-5 oscillations windows can be found for $a = 3.73817237$ but also for $a = 3.906$ and $a = 3.99026$, while period-7 oscillations can be found in nine different ranges of a.

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