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Existence of multi-pulse discrete breathers in Fermi-Pasta-Ulam-Tsingou lattices

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Abstract— Discrete breathers are spatially localized periodic solutions in nonlinear lattices. We have proved the existence of multi-pulse discrete breathers in strong localization regime in one-dimensional infinite Fermi-Pasta-Ulam-Tsingou (FPUT) lattices with even interaction potentials. Exponential localization of those discrete breathers also have been proved. The multi-pulse discrete breather consists of an arbitrary number of the odd-like and/or even-like primary discrete breathers located separately on the lattice. The existence of odd and even symmetric single-pulse discrete breathers is included as particular cases.

1. Introduction

Spatially localized excitation can exist ubiquitously in nonlinear space-discrete dynamical systems, and it has attracted great interest. Takeno *at al.* found a time-periodic and spatially localized mode in the Fermi-Pasta-Ulam-Tsingou (FPUT) lattice based on approximate analytical calculation [1, 2]. A few years later, a different type of the localized mode was also found for the FPUT lattice [3]. The localized mode is called *discrete breather* (DB) or *intrinsic localized mode*. Considerable progress has been achieved in understanding the nature of DB so far (e.g., [4, 5] and references therein).

The DBs are time-periodic and spatially localized solutions of the equations of motion of nonlinear lattices. From the mathematical point of view, a fundamental issue is their existence. The first existence proof of DB was given for the nonlinear Klein-Gordon lattice, based on the *anti-continuous* limit [6]. This limit is a useful concept, and existence proofs based on it have been given for other lattice models [7, 8, 9]. Stability results for DBs also has been given near the limit [10, 11, 12, 13, 14, 15].

The FPUT lattice is one of the fundamental lattice models in physics, to which the anti-continuous limit approach is not applicable. Normalized spatial profiles of two types of single-pulse DBs, which are called odd and even modes, are approximately given by $(\dots, 0, -1/2, 1, -1/2, 0, \dots)$ [1, 2] and $(\dots, 0, -1, 1, 0, \dots)$ [3] in the regime of strong localization, respectively. In addition, multi-pulse DBs also have been found numerically [16].

For the FPUT model, the first existence proof of the odd and even DBs was given in a particular case of the homogeneous potential [17]. In the more general case of non-homogeneous potentials, an existence proof was given for weakly localized odd and even DBs by using a center manifold reduction technique [18]. The existence of strongly localized odd and even DBs has been proved recently for finite FPUT lattices [19]. The present theorem is its extension to the case of infinite FPUT lattices. In addition to the single-pulse DBs, the theorem ensures the existence of infinitely many multi-pulse DBs and their exponential localization.

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2. Lattice model

We consider the one-dimensional infinite FPUT lattice described by the equations of motion

$$\dot{q}_i = p_i, \quad \dot{p}_i = V'(q_{i+1} - q_i) - V'(q_i - q_{i-1}), \quad i \in \mathbb{Z}, \quad (1)$$

where $q_i,\ p_i \in \mathbb{R}$ are the position and momentum of ith particle of unit-mass, respectively, and V is a potential function of the nearest neighbor interaction. Equation (1) forms an infinite system of ordinary differential equations, which has the Hamiltonian

$$H = \sum_{i=-\infty}^{\infty} \frac{1}{2} p_i^2 + \sum_{i=-\infty}^{\infty} V(q_{i+1} - q_i).$$

Equation (1) is derived by $\dot{q}_i = \partial H/\partial p_i$ and $\dot{p}_i = -\partial H/\partial q_i$.

Let $X \in \mathbb{R}$ and $\mu \in \mathbb{R}$ be a parameter. We assume the interaction potential V to be defined by

$$V(X) = \mu W(X) + \frac{1}{k} X^k, \quad (2)$$

where:

- (P1) $k \geq 4$ is an even integer;
- (P2) $W(X) : \mathbb{R} \to \mathbb{R}$ is a C^3 function of X;
- (P3) $W(X) = W(-X)$ for $\forall X \in \mathbb{R}$ and $W(0) = 0$.

Condition (P3) ensures that $V(X)$ is an even function of X. A typical non-homogeneous potential often used in the literature is a polynomial potential. Equation (2) reduces to this case when W is an even polynomial of order less than k, i.e., $W(X) = \sum_{r=1}^{k/2-1} \kappa_{2r} X^{2r}$.

3. Sequence and function spaces

Let $l^2(\mathbb{Z})$ be the Hilbert space of square-summable two-sided real-valued sequences endowed with the norm $\|\boldsymbol{x}\|_{l^2} = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle}$, which is derived from the inner product $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \sum_{i\in\mathbb{Z}} x_i y_i$, where $\boldsymbol{x} = (x_i)_{i\in\mathbb{Z}}$ and $\boldsymbol{y} = (y_i)_{i\in\mathbb{Z}}$. Let $\boldsymbol{q} = (q_i)_{i\in\mathbb{Z}} \in l^2(\mathbb{Z})$ and $\boldsymbol{p} = (p_i)_{i\in\mathbb{Z}} \in l^2(\mathbb{Z})$. We choose the phase space of Eq. (1) as $\Omega = l^2(\mathbb{Z}) \times l^2(\mathbb{Z})$ with the norm

$$\|\boldsymbol{z}\|_\Omega = \sqrt{\|\boldsymbol{q}\|_{l^2}^2 + \|\boldsymbol{p}\|_{l^2}^2},$$

where $\boldsymbol{z} = (z_i)_{i\in\mathbb{Z}} \in \Omega$ and $z_i = (q_i, p_i) \in \mathbb{R}^2$. The phase space Ω is also the Hilbert space.

Let $C_T^0(\mathbb{R};\Omega)$ be the space of T-periodic continuous functions $\boldsymbol{z}(t) : \mathbb{R} \to \Omega$ that is endowed with the norm

$$\|\boldsymbol{z}\|_0 = \sup_{t\in[0,T]} \|\boldsymbol{z}(t)\|_\Omega.$$

Let $C_T^1(\mathbb{R};\Omega)$ be the space of T-periodic continuously differentiable functions $\boldsymbol{z}(t) : \mathbb{R} \to \Omega$ that is endowed with the norm

$$\|\boldsymbol{z}\|_1 = \sup_{t\in[0,T]} \|\boldsymbol{z}(t)\|_\Omega + \sup_{t\in[0,T]} \|\dot{\boldsymbol{z}}(t)\|_\Omega.$$

Both $C_T^0(\mathbb{R};\Omega)$ and $C_T^1(\mathbb{R};\Omega)$ are Banach spaces.

Let $X_1 \subset C_T^1(\mathbb{R};\Omega)$ and $X_0 \subset C_T^0(\mathbb{R};\Omega)$ be the subspaces defined by

$$\begin{aligned} X_1 = \Big\{ &(q_i(t), p_i(t))_{i\in\mathbb{Z}}\,;\; q_i(-t) = q_i(t),\; p_i(-t) = -p_i(t),\\ &q_i(t+T/2) = -q_i(t),\; p_i(t+T/2) = -p_i(t),\; i\in\mathbb{Z} \Big\} \end{aligned}$$

and

$$\begin{aligned} X_0 = \Big\{ &(s_i(t), r_i(t))_{i\in\mathbb{Z}}\,;\; s_i(-t) = -s_i(t),\; r_i(-t) = r_i(t),\\ &s_i(t+T/2) = -s_i(t),\; r_i(t+T/2) = -r_i(t),\; i\in\mathbb{Z} \Big\}. \end{aligned}$$

Both X_1 and X_0 are Banach spaces.

In order to discuss the exponential localization property, we need to measure the amplitude of each component $z_i(t)$ of a DB solution $\boldsymbol{z}(t)$. Let $C_T^1(\mathbb{R};\mathbb{R}^2)$ be the space of T-periodic continuously differentiable functions with values in $\mathbb{R}^2$. We define the norm of $C_T^1(\mathbb{R};\mathbb{R}^2)$ as follows:

$$|z|_1 = \sup_{t\in[0,T]} |z(t)| + \sup_{t\in[0,T]} |\dot{z}(t)|,$$

where $z(t) = (q(t), p(t)) \in \mathbb{R}^2$ and $|\cdot|$ represents the Euclidean norm of $\mathbb{R}^2$. Then, it is a Banach space.

Let $b_1 \subset C_T^1(\mathbb{R};\mathbb{R}^2)$ be the subspace defined by

$$\begin{aligned} b_1 = \Big\{ &(q(t), p(t))\,;\; q(-t) = q(t),\; p(-t) = -p(t),\\ &q(t+T/2) = -q(t),\; p(t+T/2) = -p(t) \Big\}. \end{aligned}$$

This space is obtained by imposing $C_T^1(\mathbb{R};\mathbb{R}^2)$ the same type of temporal symmetries as to those of X_1. Each component of any $\boldsymbol{z} \in X_1$ can be regarded as an element of b_1.

4. Approximation of DB solutions

We describe the odd and even symmetries. Let S_O and S_E be the linear mappings $S_O, S_E : \Omega \to \Omega$ defined by

$$\begin{aligned} S_O &: (S_O\boldsymbol{z})_i = z_{-i},\quad i\in\mathbb{Z},\\ S_E &: (S_E\boldsymbol{z})_i = -z_{-(i+1)},\quad i\in\mathbb{Z}, \end{aligned}$$

where $z_i = (q_i, p_i)$ and $\boldsymbol{z} = (z_i)_{i\in\mathbb{Z}} \in \Omega$. A T-periodic solution $\boldsymbol{z}(t) \in X_1$ of Eq. (1) is said to have odd (resp. even) symmetry if it satisfies the condition $S_O\boldsymbol{z}(t) = \boldsymbol{z}(t)$ (resp. $S_E\boldsymbol{z}(t) = \boldsymbol{z}(t)$) for $\forall t \in \mathbb{R}$. The odd and even symmetric solutions have a spatial profile centered at $i = 0$ site and that centered between $i = -1$ and 0 sites, respectively.

Our existence theorem uses approximations for the spatial profiles of DB solutions. Let $\boldsymbol{a}^O = (a_i^O)_{i\in\mathbb{Z}}$ and $\boldsymbol{a}^E = (a_i^E)_{i\in\mathbb{Z}}$ be two-sided real-valued bounded sequences, each of which satisfies the condition that there are $i_1, i_2 \in \mathbb{Z}$ and $a_i^{\mathrm{x}} \neq 0$ only for $i_1 \le i \le i_2$, otherwise $a_i^{\mathrm{x}} = 0$, where the superscript x represents the sequence type O or E. We will choose $\boldsymbol{a}^O$ and $\boldsymbol{a}^E$ as in Table 1 to approximate the odd and even single-pulse DB profiles in a lattice with $\mu = 0$ and an even integer $k \ge 4$.

Approximations for the profiles of multi-pulse DBs are constructed by combining $\boldsymbol{a}^O$ and/or $\boldsymbol{a}^E$. Given $m \in \mathbb{Z}$ and a sequence $\boldsymbol{x} = (x_i)_{i\in\mathbb{Z}}$, let $\mathcal{T}_m$ be the m-site shift operator defined by

$$(\mathcal{T}_m\boldsymbol{x})_i = x_{i-m},\quad i\in\mathbb{Z}.$$

Let $n \in \mathbb{N}$, $\bar{\mathrm{x}} = (\mathrm{x}_1,\ldots,\mathrm{x}_n) \in \{O, E\}^n$, $\bar{\theta} = (\theta_1,\ldots,\theta_n) \in \{-1, 1\}^n$, and $\bar{m} = (m_1,\ldots,m_n) \in \mathbb{Z}^n$. Given $(\bar{\mathrm{x}}, \bar{\theta}, \bar{m})$, define the superposition of shifted $\boldsymbol{a}^O$ and/or $\boldsymbol{a}^E$ as follows:

$$\boldsymbol{s}_{\bar{\mathrm{x}},\bar{\theta},\bar{m}} = \sum_{j=1}^{n} \theta_j \cdot \mathcal{T}_{m_j}\boldsymbol{a}^{\mathrm{x}_j}. \tag{3}$$

For each $\mathcal{T}_{m_j}\boldsymbol{a}^{\mathrm{x}_j}$ in the sum, we define its support by

$$J(\mathcal{T}_{m_j}\boldsymbol{a}^{\mathrm{x}_j}) = \Big\{ i\in\mathbb{Z}\,;\, (\mathcal{T}_{m_j}\boldsymbol{a}^{\mathrm{x}_j})_i \neq 0 \Big\}.$$

This support is a finite set since each of $\boldsymbol{a}^O$ and $\boldsymbol{a}^E$ has only a finite number of consecutive nonzero elements. When $J(\mathcal{T}_{m_j}\boldsymbol{a}^{\mathrm{x}_j}) = \{i_1, i_1+1, \ldots, i_2\}$, let $\bar{J}(\mathcal{T}_{m_j}\boldsymbol{a}^{\mathrm{x}_j})$ be the set defined by

$$\bar{J}(\mathcal{T}_{m_j}\boldsymbol{a}^{\mathrm{x}_j}) = J(\mathcal{T}_{m_j}\boldsymbol{a}^{\mathrm{x}_j}) \cup \{i_1 - 1, i_2 + 1\},$$

which is an extended support by adding two adjacent sites of $J(\mathcal{T}_{m_j}\boldsymbol{a}^{\mathrm{x}_j})$. Using these notations, we define the extended support of $\boldsymbol{s}_{\bar{\mathrm{x}},\bar{\theta},\bar{m}}$ by

$$\bar{J}(\boldsymbol{s}_{\bar{\mathrm{x}},\bar{\theta},\bar{m}}) = \bigcup_{j=1}^{n} \bar{J}(\mathcal{T}_{m_j}\boldsymbol{a}^{\mathrm{x}_j}).$$

5. Main results

Consider the scalar differential equation

$$\ddot{\phi} + \phi^{k-1} = 0,$$

where $k \geq 4$ is an even integer. Given any $T > 0$, let $\phi(t;T)$ be its T-periodic solution with initial conditions $\phi(0;T) > 0$ and $\dot{\phi}(0;T) = 0$.

Let $l_1, l_2 \in \mathbb{Z}$, $l_1 \leq l_2$ and $D_{c,r}(l_1, l_2) \subset l^2(\mathbb{Z})$ be a closed convex subset defined by

$$D_{c,r}(l_1,l_2) = \Big\{ \boldsymbol{x}\,;\, |x_i| \leq c \text{ for } l_1 \leq i \leq l_2,\ |x_i| \leq cr^{(k-1)^{l_1-i}} \text{ for } i \leq l_1 - 1,\ |x_i| \leq cr^{(k-1)^{i-l_2}} \text{ for } i \geq l_2 + 1 \Big\},$$

where $c > 0$ and $0 < r < 1$. This subset $D_{c,r}(l_1, l_2)$ is specified by the four parameters (l_1, l_2, c, r), and the interval of x_i rapidly (super-exponentially) decreases with increasing $|i|$ in $D_{c,r}(l_1, l_2)$. The existence theorem is stated as follows. Its proof of the existence part is given in Ref. [20].

Theorem 5.1 *Suppose $V(X)$ of Eq. (2) and (P1)-(P3). Fix the value of k and choose $\boldsymbol{a}^O = (a_i^O)_{i\in\mathbb{Z}}$, $\boldsymbol{a}^E = (a_i^E)_{i\in\mathbb{Z}}$, and (c, r) as in Table 1. Let $\boldsymbol{s}_{\bar{\chi},\bar{\theta},\bar{m}}$ be a superposition given by Eq. (3) such that $\bar{J}(\mathcal{T}_{m_i}\boldsymbol{a}^{\chi_i}) \cap \bar{J}(\mathcal{T}_{m_j}\boldsymbol{a}^{\chi_j}) = \phi$ for any $i \neq j$. Let $l_1 = \min \bar{J}(\boldsymbol{s}_{\bar{\chi},\bar{\theta},\bar{m}})$ and $l_2 = \max \bar{J}(\boldsymbol{s}_{\bar{\chi},\bar{\theta},\bar{m}})$. Then, for any $T > 0$, there exists a unique $\boldsymbol{x} \in D_{c,r}(l_1, l_2)$ such that*

$$\boldsymbol{\Gamma}_0(t;T) = (\, u_i\phi(t;T),\ u_i\dot{\phi}(t;T)\,)_{i\in\mathbb{Z}}$$

with $(u_i)_{i\in\mathbb{Z}} = \boldsymbol{s}_{\bar{\chi},\bar{\theta},\bar{m}} + \boldsymbol{x}$ is a T-periodic solution of FPUT lattice (1) with $\mu = 0$. Moreover, there exist $\mu_0 > 0$ and a family $\boldsymbol{\Gamma}(t;T,\mu)$ of T-periodic solutions of FPUT lattice (1) for $\mu \in [-\mu_0, \mu_0]$ such that it is a unique continuation of $\boldsymbol{\Gamma}_0(t;T)$ in X_1 continuous with respect to μ and exponentially localized in space, i.e., there exist $K > 0$, $h > 0$, and $\rho \in (0,1)$ such that components $z_i(\mu)$ of $\boldsymbol{\Gamma}(t;T,\mu)$ satisfy

$$|z_i(\mu)|_1 \leq K \exp(h|\mu|)\,\rho^{|i|} \tag{4}$$

for all $i \in \mathbb{Z}$. If $\boldsymbol{s}_{\bar{\chi},\bar{\theta},\bar{m}} = \boldsymbol{a}^O$ (resp. $\boldsymbol{a}^E$), then $\boldsymbol{\Gamma}(t;T,\mu)$ has odd (resp. even) symmetry.

$k = 4$	$a_0^O = 0.3762$, $a_{\pm1}^O = -0.1968$, $a_{\pm2}^O = 0.00867$, $a_0^E = -a_{-1}^E = 0.32301$, $a_1^E = -a_{-2}^E = -0.053551$, $a_i^{O,E} = 0$ (otherwise) $(c,r) = (\,0.0015,\ 0.02\,)$
$k = 6$	$a_0^O = 0.50566$, $a_{\pm1}^O = -0.25391$, $a_{\pm2}^O = 0.00108$, $a_0^E = -a_{-1}^E = 0.4166$, $a_1^E = -a_{-2}^E = -0.015$, $a_i^{O,E} = 0$ (otherwise) $(c,r) = (\,1.2\times10^{-4},\ 9\times10^{-4}\,)$
$k = 8$	$a_0^O = 0.55502$, $a_{\pm1}^O = -0.27764$, $a_0^E = -a_{-1}^E = 0.44484$, $a_1^E = -a_{-2}^E = -0.00365$, $a_i^{O,E} = 0$ (otherwise) $(c,r) = (\,2\times10^{-4},\ 8\times10^{-4}\,)$
$k = 10$	$a_0^O = 0.58111$, $a_{\pm1}^O = -0.29057$, $a_0^E = -a_{-1}^E = 0.45839$, $a_1^E = -a_{-2}^E = -9.1\times10^{-4}$, $a_i^{O,E} = 0$ (otherwise) $(c,r) = (\,3\times10^{-5},\ 2\times10^{-4}\,)$
$k = 12$	$a_0^O = 0.59730$, $a_{\pm1}^O = -0.29865$, $a_0^E = -a_{-1}^E = 0.46649$, $a_i^{O,E} = 0$ (otherwise) $(c,r) = (\,4\times10^{-4},\ 9\times10^{-4}\,)$
$k \geq 14$	$a_0^O = 2\times3^{-(k-1)/(k-2)}$, $a_{\pm1}^O = -3^{-(k-1)/(k-2)}$, $a_0^E = -a_{-1}^E = (1+2^{k-1})^{-1/(k-2)}$, $a_i^{O,E} = 0$ (otherwise) $(c,r) = (\,3\,(1+2^{k-1})^{-(k-1)/(k-2)},\ 1\times10^{-3}\,)$

Table 1

We outline our proof of existence of the DB solution $\boldsymbol{\Gamma}(t;T,\mu)$. The proof consists of two steps. In the first step, we consider the homogeneous potential FPUT lattice which is described by Eq. (1) with the potential $V(X) = X^k/k$, i.e., $\mu = 0$ in Eq. (2). In this particular lattice, it is possible to find a DB solution in the form $\boldsymbol{q} = \boldsymbol{u}\phi(t;T)$, where $\boldsymbol{u} = (u_i)_{i\in\mathbb{Z}} \in l^2(\mathbb{Z})$ is a time-independent constant vector describing the spatial profile of the solution. Given an approximate profile $\boldsymbol{s}_{\bar{\chi},\bar{\theta},\bar{m}}$, the vector $\boldsymbol{u}$ is determined by solving a set of infinite algebraic equations in a neighborhood of $\boldsymbol{s}_{\bar{\chi},\bar{\theta},\bar{m}}$ with use of Banach's fixed point theorem. The obtained solution is $\boldsymbol{\Gamma}_0(t;T)$. That is, we have $\mathcal{F}(\boldsymbol{z}_0, 0) = \boldsymbol{0}$, where $\boldsymbol{z}_0 := \boldsymbol{\Gamma}_0(t;T)$. In the second step, we consider the non-homogeneous potential FPUT lattice, i.e., $\mu \neq 0$ in Eq. (2). Let $D\mathcal{F}(\boldsymbol{z},\mu)$ be the Fréchet derivative of $\mathcal{F}(\boldsymbol{z},\mu)$ with respect to $\boldsymbol{z}$. It is possible to prove that $D\mathcal{F}(\boldsymbol{z},\mu)$ is invertible at $(\boldsymbol{z}_0, 0)$. The solution $\boldsymbol{z}_0$ is uniquely continued to a non-homogeneous potential lattice for small $\mu \neq 0$ in X_1 by applying the implicit function theorem to $\mathcal{F}(\boldsymbol{z},\mu) = \boldsymbol{0}$. Then, $\boldsymbol{z}(\mu) = \boldsymbol{\Gamma}(t;T,\mu)$ is obtained for μ small enough as the function such that $\mathcal{F}(\boldsymbol{z}(\mu),\mu) = \boldsymbol{0}$ and $\boldsymbol{z}(0) = \boldsymbol{z}_0$.

6. Strategy of the proof of Theorem 5.1

The DB solutions of Eq. (1) are formulated as zeros of the following μ-dependent operator $\mathcal{F}$, i.e., $\boldsymbol{z}$ such that $\mathcal{F}(\boldsymbol{z},\mu) = \boldsymbol{0}$.

Definition 6.1 *Let $\mathcal{F} : X_1 \times \mathbb{R} \to X_0$ be the operator defined as follows:*

$$\mathcal{F}(\boldsymbol{z},\mu) = \boldsymbol{w}, \qquad (\boldsymbol{z},\mu) \in X_1 \times \mathbb{R},$$

where, denoting $\boldsymbol{z} = (q_i, p_i)_{i\in\mathbb{Z}}$ and $\boldsymbol{w} = (s_i, r_i)_{i\in\mathbb{Z}}$, we have

$$\begin{aligned} s_i &= \dot{q}_i - p_i, \\ r_i &= \dot{p}_i - V'(q_{i+1} - q_i) + V'(q_i - q_{i-1}), \end{aligned}$$

with V given by Eq. (2).

The implicit function theorem tells that $z(\mu)$ satisfies the following differential equation defined in X_1:

$$\frac{dz}{d\mu} = -D\mathcal{F}^{-1}(z,\mu) \cdot \mathcal{F}_\mu(z,\mu), \qquad (5)$$

where $\mathcal{F}_\mu(z,\mu)$ is the Fréchet derivative of $\mathcal{F}$ with respect to μ. The linear operator $D\mathcal{F}(z,\mu)$ has the block tridiagonal form. MacKay and Aubry proved a lemma that if a block tridiagonal operator is invertible then elements of the inverse matrix decay exponentially with distance from the diagonal [6]. The exponential localization of $z(\mu)$ is proved by using a slightly modified version of the lemma.

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