Analysis of a Relaxation Oscillator Driven by a Square Wave External Force

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Abstract—We consider the relaxation oscillator which contains time-variant threshold. The time-variant threshold is driven by a square wave external force. We analyze synchronization phenomena of the system against the external force. We clarify that the synchronization phenomena are depended on the value of the threshold.

1. Introduction

Synchronization phenomena can be observed universally in nature, also they are very interesting phenomena. For example, we can observe a phase synchronization behavior in Southeast Asian fireflies [1]. A circadian rhythm is one example of such synchronization phenomena. The circadian rhythm is that plants and animals are driven by the solar cycle. The population synchronization behavior caused by a common external force has also been reported [2].

These synchronization phenomena can be simulated by using various oscillator systems. Various kinds of oscillator systems have been proposed. In this article, we pay attention to a relaxation oscillator since the structure of the system is simple and the behavior can be analyzed rigorously. For such relaxation oscillator systems, the synchronization phenomena of the coupled relaxation oscillators have been analyzed [3][4]. Kohari et.al. have been analyzed the characteristic of the attractors when the system is driven by an external force [5]. In this article, we pay attention to the following relaxation system[6][7].

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -x(t) + y(t) \tag{1}$$

$$y(t) = h(t, x(t)) = \begin{cases} 1 & x(t) < S(t) \\ -1 & x(t) > W(t) \end{cases} \tag{2}$$

,where $x(t)$ denotes a state variable of the relaxation oscillator, $y(t)$ denotes an output. $h(t, x(t))$ represents a bipolar hysteresis as shown in Fig. 1. The threshold of the bipolar hysteresis is driven by a square waveform as described by

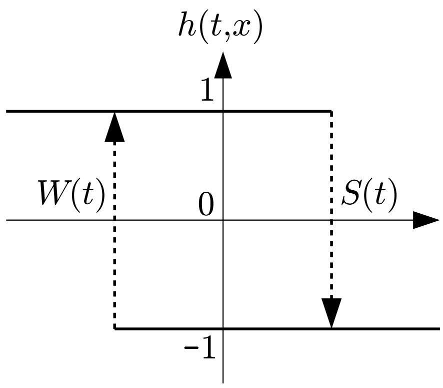


Figure 1: Hysteresis function $h(t, x)$

Eqs. (3), (4), (5) and (6).

$$S(t) = \begin{cases} h = \alpha + \beta & 0 \le t < \frac{T}{2} \\ l = \alpha - \beta & \frac{T}{2} \le t < T \end{cases} \tag{3}$$

$$W(t) = \begin{cases} -l = -\alpha + \beta & 0 \le t < \frac{T}{2} \\ -h = -\alpha - \beta & \frac{T}{2} \le t < T \end{cases} \tag{4}$$

$$S(t + T) = S(t) \tag{5}$$

$$W(t + T) = W(t) \tag{6}$$

,where α means an original threshold of the relaxation oscillator. β and T denote an amplitude of the square wave and a period, respectively.

Namely, the threshold of the bipolar hysteresis is varied alternately by the external force.

2. Definition of return map

In order to analyze the dynamics of the relaxation oscillator system with the square wave external force, we derive a return map from the system. The output of this system is switched when the state variable hits the threshold. The switching points on the threshold region which can be classified into three intervals I_0, I_1 and I_2 as shown in Fig. 2(b).

$$\begin{cases} I_0 = \{(t, x) | 0 \le t < T/2,\ x = h\} \\ I_2 = \{(t, x) | t = T/2,\ l < x \le h\} \\ I_1 = \{(t, x) | T/2 \le t \le T\ \ x = l\} \end{cases} \tag{7}$$

These intervals are described as

$$g(t,x(t)) = \begin{cases} \frac{1}{L}t & 0 \le t < \frac{T}{2},\ x(t) = h \\ \frac{1}{L}(t+h-x(t)) & t = \frac{T}{2},\ l \le x(t) \le h \\ \frac{1}{L}(t+h-l) & \frac{T}{2} \le t < T,\ x(t) = l \end{cases} \quad (8)$$

$$L = T + h - l \quad (9)$$

Note that the domain has a periodic property described as

$$g(t,x) = g(t+T,x) \quad (10)$$

$$g(t,x) = g(t+\frac{T}{2}, -x) \quad (11)$$

We normalize this domain into the [0,1] interval. By using this normalized domain, we define a return map.

The relation between any initial value $(t_0, x_0) \in I_0 \cup I_1 \cup I_2$ and the corresponding switching point (t_1, x_1) is given as the followings:

$$(t_1, x_1) = f(t_0, x_0) \quad (12)$$

$$= \begin{cases} (t_l, -l) & t_0 < t_l \text{ and } t_l \bmod T < T/2 \\ (t_h, -h) & t_0 < t_l \text{ and } t_l \bmod T > T/2 \\ & \text{and } t_h \bmod T > T/2 \\ & \text{and } t_h - t_l < T/2 \\ & \text{or } t_0 > t_l \text{ and } t_h \bmod T > T/2 \\ & \text{and } t_h - t_0 < T/2 \\ (t_v, z) & \text{ofhterwise} \end{cases} \quad (13)$$

$$t_l = t_0 + \ln((1+x_0)/(1-l)) \quad (14)$$

$$t_h = t_0 + \ln((1+x_0)/(1-h)) \quad (15)$$

$$t_v = \begin{cases} T - (t_l \bmod T) + t_l & t_0 < t_l \\ T - (t_0 \bmod T) + t_0 & t_0 > t_l \end{cases} \quad (16)$$

$$z = (1-l)\exp(-t_v) - 1 \quad (17)$$

Using above relationship, we define a return map to analyze the dynamics of the system.

$$X_{n+1} = F(X_n) \quad (18)$$

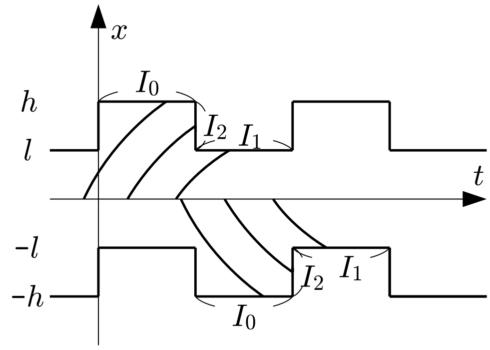
$$F = g \cdot f \cdot g^{-1} \ : [0,1] \to [0,1] \quad (19)$$

2.1. Switching Ratio

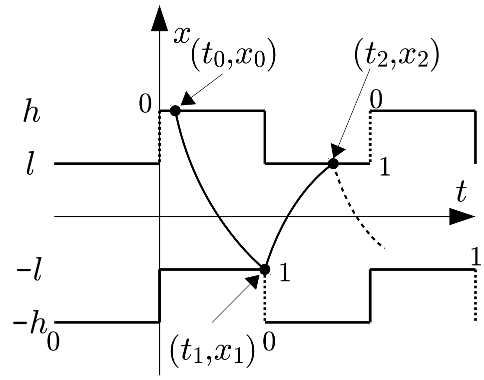
The system contains various kinds synchronization phenomena for external forces. In order to classify such synchronization phenomena, we apply a switching ratio γ which is defined as the following.

$$\gamma = \frac{\text{the number of switchings of } h(t,x)}{\text{the number of switchings of } S(t)} \quad (20)$$

This switching ratio denotes the ratio of the number of switching times of the oscillator to the number of switching times of the external force.



(a) Switching area



(b) (t_n, x_n) series

Figure 2: State of the switching

2.2. Symbol Series

In order to classify the periodic time series of the output, we define a symbol series. First, the domain of the return map is consisted of three regions. We assign each region to a symbol which is expressed by ternary numeral as the follow.

$$\omega(X_n) = \begin{cases} 0 & X_n \in F(I_0) \\ 1 & X_n \in F(I_1) \\ 2 & X_n \in F(I_2) \end{cases} \quad (21)$$

By using the expression of Eq. (21), any periodic time series is expressed by the ternary numeral symbol series.

3. Analysis

In this article, we pay attention to the case where the period of the external force is long. Specifically, we consider the case where the following condition is satisfied.

$$2\ln((1+l)/(1-l)) < T \quad (22)$$

As parameters are satisfied the condition (22), we analyze the switching rate γ by numerical simulations in the case where the amplitude β and the period T of the external force are varied. Figures 3 and 4 represent the numerical simulation results. Figure 3 shows the result of the case of $\alpha = 0.5$, and Fig. 4 shows the result of the case of $\alpha = 0.4$.

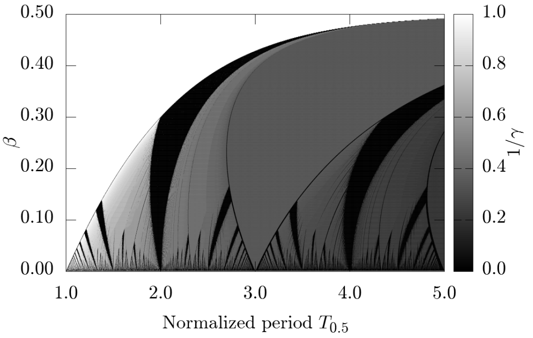


Figure 3: $\alpha = 0.5$

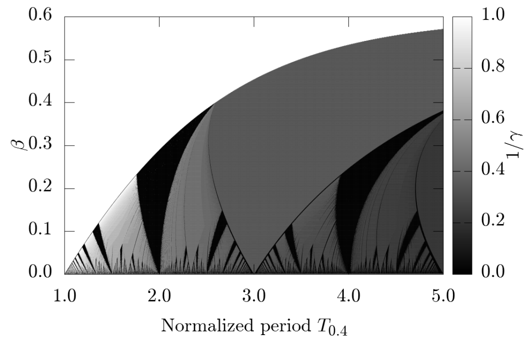


Figure 4: $\alpha = 0.4$

In these figures, the horizontal axis denotes the normalized frequency which is normalized by the oscillation frequency of $\beta = 0.0$. Each color corresponds to the switching ratio. These results exhibit complicated structure of the synchronization. In the case of Fig. 4, $\beta = 0.4$ is a critical parameter value for the classification of the synchronization phenomena. This critical point is due to a phenomenon varies depending on the sign of the parameter l. Such trend is not observed in the result of Fig. 3.

3.1. 1:1 Schronization

In this subsection, we consider the case where the output is synchronized with 1:1 against the external force. This situation corresponds to the state of $\gamma = 1.0$. In this case, the period of the relaxation oscillator is equivalent to the period of the external force. Figure 5 shows an example waveform of the oscillation. The phenomena can be ob-

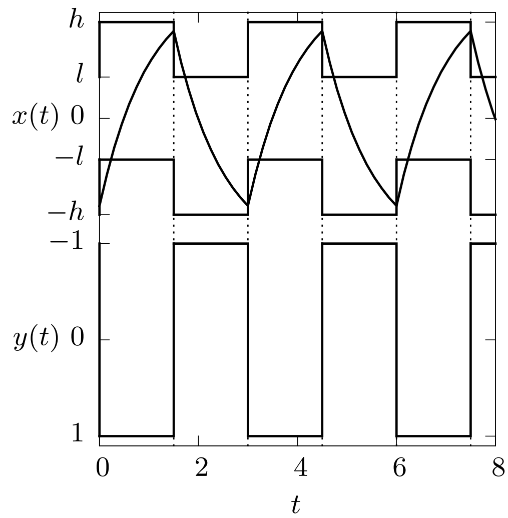


Figure 5: Examples of oscillation waveform
$\alpha = 0.5,\ \beta = 0.2,\ T = 3.0$

served when the following conditions are satisfied.

$$T_a < T < T_b \tag{23}$$

$$T_a = \begin{cases} 2\ln\left(\frac{1+l}{1-l}\right) & 0 < l \\ 0 & l \le 0 \end{cases} \tag{24}$$

$$T_b = \begin{cases} 2\ln\left(\frac{1+h}{1-h}\right) & h < 1 \\ \infty & 1 \le h \end{cases} \tag{25}$$

The synchronization region is given by the values of l and h. The lower limit period T_a is depended on the parameter l, and the upper limit period T_b is depended on the parameter h. If the parameter l becomes a small, the lower limit period T_a becomes a short. If the parameter h becomes a large, the upper limit period T_b becomes a long. If l is smaller than 0, the short period must be synchronized. On the other hand, if h is larger than 1, the long period must be synchronized. Namely, in the case of $l < 0$ and $h > 1$, the system is synchronized to arbitrary period T of the external force.

3.2. $\{2(01)^m\}$ Synchronization

In this subsection, we consider the case where the system exhibits $2(01)^m$ synchronization. The time series of the synchronization attractor corresponds to the ternary symbol expression $2(01)^m$. In this case, the switching rate γ is given as the following.

$$\frac{1}{\gamma} = \frac{1}{2m+1} \tag{26}$$

This synchronization patterns is one of the typical oscillation pattern when the period of the external force is long. The synchronization region is different on the sign of the parameter l.

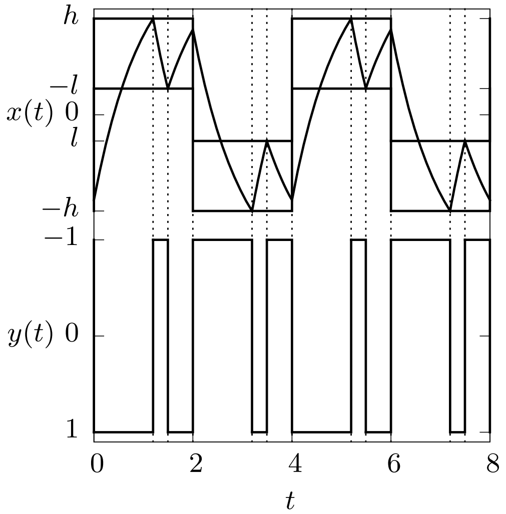


Figure 6: Examples of oscillation waveform
$\alpha = 0.2,\ \beta = 0.35,\ T = 4.0$

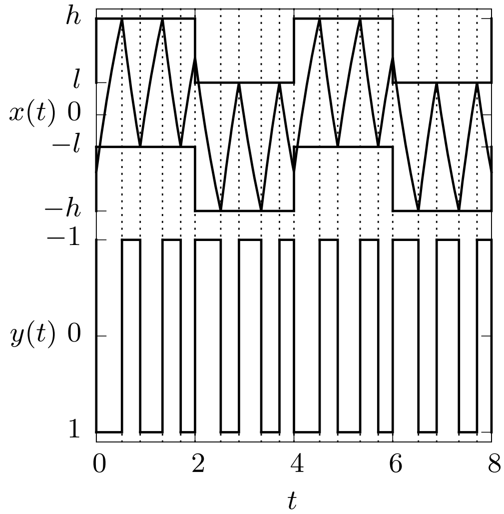


Figure 7: Examples of oscillation waveform
$\alpha = 0.2,\ \beta = 0.1,\ T = 4.0$

3.2.1. $l \le 0 < h < 1$

If $l \le 0 < h < 1$ is satisfied, the switching point in the interval I_2 is greater than $-l$, and smaller than h. In this case, the system oscillates as shown in Fig. 6. Figure 6 indicates that the switching point is existed in the interval I_2 every half period. Therefore, the system emerges $2(01)^m$ synchronization only if the following conditions are satisfied.

$$l \le 0 < h < 1 \tag{27}$$

$$T'_{c,m} < T < T'_{d,m} \tag{28}$$

$$T'_{c,m} = 2m \ln\left(\frac{1+h}{1-h}\right) + 2(m-1)\ln\left(\frac{1+l}{1-l}\right) \tag{29}$$

$$T'_{d,m} = 2(m+1) \ln\left(\frac{1+h}{1-h}\right) + 2m\ln\left(\frac{1+l}{1-l}\right) \tag{30}$$

In this case, $T'_{d,m} = T'_{c,m+1}$ is satisfied. This synchronization region becomes wide when α is small. This situation is corresponded to the region of $\beta > 0.4$ in Fig. 4.

3.2.2. $0 < l < h < 1$

If $0 < l < h < 1$ is satisfied, the switching point in the interval I_2 is greater than l, and smaller than h. An example of the oscillation waveform is shown in Fig. 7. In this case, the condition of $2(01)^m$ synchronization is described as the followings:

$$0 < l < h < 1 \tag{31}$$

$$T_{c,m} < T < T_{d,m} \tag{32}$$

$$T_{c,m} = 2m \ln\left(\frac{1+h}{1-h}\right) + 2(m+1)\ln\left(\frac{1+l}{1-l}\right) \tag{33}$$

$$T_{d,m} = 2(m+1) \ln\left(\frac{1+h}{1-h}\right) + 2m\ln\left(\frac{1+l}{1-l}\right) \tag{34}$$

In this case, $T'_{d,m} < T'_{c,m+1}$ is satisfied.

4. Conclusions

We considered the relaxation oscillator which contains time-variant threshold. The time-variant threshold is driven by the square wave external force. We analyzed synchronization phenomena of the system against the external force. In this article, we paid attention to the case where the external force has long period.

Consequently, we clarified that the synchronization phenomena is depended on the value of the threshold.

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