Amplitude Death in High-dimensional Map Networks with Connection Delays

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Abstract—This paper investigates amplitude death in high-dimensional map networks with connection delays. A stability analysis shows that death never occurs if the connection delay is zero or the fixed point has the so-called odd-number property. In addition, the robust control theory, especially a concept of convex direction, simplifies the systematic procedure for designing the connection parameters for inducing death. The designed parameters induce death for any topology and any number of maps. In order to confirm our analytical results, some numerical examples are provided.

1. Introduction

It is well known that homogeneous steady states in coupled oscillators can be stabilized by diffusive connections with time delays [1]. This stabilization phenomenon, known as *amplitude death*, has been investigated from viewpoints of physics [2, 3, 4] and engineering [5, 6]. Most of these studies deal with amplitude death in coupled *continuous-time* oscillators. The stability analysis of death in such oscillators is not easy because the dimension of their dynamics becomes *infinite*. As a consequence, these studies are forced to employ some particular techniques for analyzing time-delay systems. In contrast, the stability analysis in coupled *discrete-time* maps is easy due to its *finite* dimension; there is no need to employ such techniques. However, to our knowledge, only few studies have been made at amplitude death in the coupled discrete-time maps.

The first observation of amplitude death in coupled maps was reported in our previous study [7]. The study dealt with *a pair of high-dimensional* maps coupled by a delay connection, where some analytical results were obtained: no-delay connection never induces death; the well known odd-number property still remains; death never occurs in a pair of one-dimensional maps [7]. Further, amplitude death in *one-dimensional* map networks with uniform delays [8] and non-uniform delays [9, 10, 11] were reported. However, there is no report on death in *high-dimensional map networks* with connection delays due to its difficulty of stability analysis.

The present paper analyzes the stability of death in high-dimensional map networks with uniform delay connections. We present that the similar results reported in our previous study [7] are valid even for high-dimensional map networks. Furthermore, we suggest that the robust control theory gives us a simple systematic procedure for designing the connection parameters. The designed parameters are valid for any topology and any number of maps. Some numerical examples are employed to confirm our analytical results. The present work can be regarded as an extension of study [7] for networks and that of study [8] for high-dimensional maps.

2. Map networks

Consider N identical maps,

$$\left\{ \begin{array}{rcl} \boldsymbol{x}_i(n+1) & = & \boldsymbol{F}\left[\boldsymbol{x}_i(n)\right] + \boldsymbol{b}u_i(n), \\ y_i(n) & = & \boldsymbol{c}\boldsymbol{x}_i(n), \end{array} \right. \tag{1}$$

for $i = 1, \ldots, N$. Here $\boldsymbol{x}_i(n) \in \boldsymbol{R}^m$ is the m-dimensional state variable of map i at time n. The input and output signals are $u_i(n) \in \boldsymbol{R}$ and $y_i(n) \in \boldsymbol{R}$, respectively. The nonlinear function $\boldsymbol{F} : \boldsymbol{R}^m \to \boldsymbol{R}^m$ has the fixed point $\boldsymbol{x}^* : \boldsymbol{x}^* = \boldsymbol{F}\left[\boldsymbol{x}^*\right]$. $\boldsymbol{b} \in \boldsymbol{R}^m$ and $\boldsymbol{c} \in \boldsymbol{R}^{1\times m}$ are the input and output vectors, respectively. The input signal is given by

$$u_i(n) = k\left[\left\{\sum_{j=1}^{N} \frac{\varepsilon_{ij}}{d_i} y_j(n-\tau)\right\} - y_i(n)\right], \tag{2}$$

where $\tau \in \boldsymbol{Z}_+$ denotes the delay time and $k \in \boldsymbol{R}$ is the coupling strength. $\varepsilon_{ij} \in \{0, 1\}$ describes the following situation: if map i and map j are connected, then $\varepsilon_{ij} = \varepsilon_{ji} = 1$, otherwise $\varepsilon_{ij} = \varepsilon_{ji} = 0$; $\varepsilon_{ii} = 0$, $\forall i$. The number of connections to map i is denoted by $d_i := \sum_{j=1}^{N} \varepsilon_{ij}$.

Maps (1) with connection (2) have the steady state,

$$\left[\boldsymbol{x}_1(n)^T \quad \cdots \quad \boldsymbol{x}_N(n)^T\right]^T = \left[\boldsymbol{x}^{*T} \quad \cdots \quad \boldsymbol{x}^{*T}\right]^T, \tag{3}$$

which is the spatially uniform equilibrium solution. The linearized map around state (3) is given by

$$\boldsymbol{v}_i(n+1) = (\boldsymbol{A} - k\boldsymbol{bc})\,\boldsymbol{v}_i(n) + k\boldsymbol{bc}\sum_{j=1}^{N}\frac{\varepsilon_{ij}}{d_i}\boldsymbol{v}_j(n-\tau),$$

for $i = 1, \ldots, N$ with $\boldsymbol{v}_i(n) := \boldsymbol{x}_i(n) - \boldsymbol{x}^*$. Assume that the Jacobi matrix $\boldsymbol{A} := \{\partial \boldsymbol{F}(\boldsymbol{x})/\partial \boldsymbol{x}\}_{\boldsymbol{x}=\boldsymbol{x}^*}$ is unstable (i.e., $\boldsymbol{x}^*$ is unstable) throughout this paper. This linear map can be rewritten as

$$\begin{aligned}\boldsymbol{V}(n+1) = \left[\boldsymbol{I}_N \otimes (\boldsymbol{A} - k\boldsymbol{bc})\right]\boldsymbol{V}(n)\\ + (\boldsymbol{E} \otimes k\boldsymbol{bc})\boldsymbol{V}(n-\tau),\end{aligned} \quad (4)$$

where

$$\boldsymbol{V}(n) := \begin{bmatrix}\boldsymbol{v}_1(n)\\ \vdots \\ \boldsymbol{v}_N(n)\end{bmatrix},\ \boldsymbol{E} := \begin{bmatrix}\varepsilon_{11}/d_1 & \cdots & \varepsilon_{1N}/d_1\\ \vdots & \ddots & \vdots\\ \varepsilon_{N1}/d_N & \cdots & \varepsilon_{NN}/d_N\end{bmatrix}.$$

Let us remember that the local stability of state (3) is equivalent to that of mN-dimensional linear map (4).

3. Stability analysis

This section will consider the stability of linear map (4), which is governed by the characteristic polynomial,

$$\begin{aligned}\bar{G}(z) := \det\left[z\boldsymbol{I}_{mN} - \boldsymbol{I}_N \otimes (\boldsymbol{A} - k\boldsymbol{bc})\right.\\ \left.-(\boldsymbol{E} \otimes k\boldsymbol{bc})z^{-\tau}\right].\end{aligned} \quad (5)$$

Since matrix $\boldsymbol{I}_N - \boldsymbol{E}$ can be diagonalized with a diagonal transformation matrix $\boldsymbol{T}$ [12],

$$\boldsymbol{T}^{-1}(\boldsymbol{I}_N - \boldsymbol{E})\boldsymbol{T} = \mathrm{diag}(\rho_1, \ldots, \rho_N), \quad (6)$$

$$0 = \rho_1 \le \rho_2 \le \cdots \le \rho_N \le 2, \quad (7)$$

we have $\bar{G}(z) := \prod_{q=1}^{N}\bar{g}(z, \rho_q)$, where $\bar{g}(z,\rho) := d(z) + n(z)k\left\{1 - (1-\rho)z^{-\tau}\right\}$. Here the polynomials $d(z)$ and $n(z)$ are denoted by

$$\frac{n(z)}{d(z)} := \boldsymbol{c}(z\boldsymbol{I}_m - \boldsymbol{A})^{-1}\boldsymbol{b} = \frac{\boldsymbol{c}\,\mathrm{adj}(z\boldsymbol{I}_m - \boldsymbol{A})\boldsymbol{b}}{\det\left[z\boldsymbol{I}_m - \boldsymbol{A}\right]}.$$

Remark that the stability of $\bar{G}(z)$ is equivalent to that of

$$G(z) := \prod_{q=1}^{N} g(z, \rho_q), \quad (8)$$

$$g(z,\rho) := z^\tau d(z) + n(z)k\left(z^\tau - 1 + \rho\right). \quad (9)$$

It should be noted that $n(z)/d(z)$ is a transfer function of map (1), at the fixed point $\boldsymbol{x}^*$, from the input $u_i(n)$ to the output $y_i(n)$. This paper will discuss the stability of $G(z)$ below.

Now we consider some properties of the stability of steady state (3). In order to derive these properties, we should understand the following fact: a sufficient condition for steady state (3) to be *unstable* is that $g(z,0)$ is *unstable*, since $G(z)$ inevitably includes $g(z,\rho_1)$ with $\rho_1 = 0$. From this fact, we can obtain the following lemmas.

Lemma 1. *Steady state (3) in maps (1) with connection (2) is unstable for any topology $\boldsymbol{E}$ and any coupling strength k, if the connection delay time τ is zero (i.e., no-delay connection).*

Proof. The characteristic polynomial $g(z,0)$ with $\tau = 0$, $g(z,0) = d(z)$, does not depend on $\boldsymbol{E}$ and k. The assumption, $\boldsymbol{A}$ is unstable, suggests that $d(z)$, the characteristic polynomial of $\boldsymbol{A}$, is also unstable. Thus, we conclude that the characteristic polynomial $G(z) = 0$ with $\tau = 0$, which includes $g(z,0) = d(z) = 0$, has the unstable roots. □

Lemma 2. *Steady state (3) in maps (1) with connection (2) is unstable for any topology $\boldsymbol{E}$, any coupling strength k, and any delay time τ, if the Jacobi matrix $\boldsymbol{A}$ has the odd number property (i.e., $\boldsymbol{A}$ has an odd number of real eigenvalues greater than 1).*

Proof. The characteristic polynomial $g(z,0)$ with $z = 1$ is $g(1,0) = d(z)$, which does not depend on $\boldsymbol{E}$, k, and τ. If the characteristic equation of $\boldsymbol{A}$, that is, $d(z) = 0$, has an odd number of real roots greater than 1, we have $d(1) < 0$. Thus, we see that if $\boldsymbol{A}$ has the odd number property, $g(z,0) = 0$ has at least one real root $z > 1$. □

These unstable properties for $N = 2$ were derived in study [7]; hence, Lemmas 1 and 2 imply that these properties hold even for networks with $N > 2$.

In addition, we can easily provide the following property: steady state (3) in one-dimensional ($m = 1$) maps (1) coupled by connection (2) with bipartite topologies ($\rho_N = 2$) is unstable for any coupling strength k and τ. This property has been already given in previous studies [7, 8].

4. Design of connection parameters

This section will propose a systematic procedure for designing the connection parameters k and τ, which are valid for any topology $\boldsymbol{E}$ and any number of maps N. We will show that the parametric approach in robust control theory allows us to derive the procedure.

Let us consider a one parameter family of polynomials,

$$L(z) := \{g(z,\rho) : \rho \in [0,2]\},$$

known as a segment of polynomials. Note that the parameter ρ belongs to the interval $[0,2]$ due to condition (7) for any $\boldsymbol{E}$ and N. Thus, our design problem can be reduced to a choice of k and τ such that $L(z)$ is stable. All the coefficients of $g(z,\rho)$ defined in Eq. (9) are described by affine functions of ρ; then $L(z)$ can be expressed by

$$L(z) = \{g(z,0) + \mu\hat{g}(z) : \mu \in [0,1]\}, \qquad (10)$$

where $\hat{g}(z) := g(z,2) - g(z,0)$ presents the direction of the segment.

It is well known in the robust control theory [13, 14] that $L(z)$ is a stable segment if the following three conditions hold (see Appendix A): (a) $g(z,0)$ is stable; (b) $g(z,2)$ is stable; (c) $\hat{g}(z)$ is a convex direction. The conditions (a) and (b) are easily checked by popular stability criteria, such as the Jury stability test [15].

Now we will explain how to check condition (c). Substituting polynomials (9) with $\rho = 0$ and 2 into $\hat{g}(z) := g(z,2) - g(z,0)$, we have

$$\hat{g}(z) = 2kn(z). \qquad (11)$$

The direction $\hat{g}(z)$ and the definition of convex directions (see Appendix A) yield the following simple condition.

Lemma 3. *$\hat{g}(z)$ is a convex direction if the following inequality holds:*

$$\frac{1}{n_r^2 + n_i^2}\left(n_r\frac{\mathrm{d}n_i}{\mathrm{d}\theta} - n_i\frac{\mathrm{d}n_r}{\mathrm{d}\theta}\right) \leq \frac{m}{2} \qquad (12)$$

for $\theta \in (0,\pi)$, where $n\left(e^{j\theta}\right) := n_r(\theta) + jn_i(\theta)$.

Proof. It is easy to deduce from Theorem 2 in Appendix A that a sufficient condition for $\hat{g}(z)$ to be a convex direction is

$$\frac{\partial \arg\left\{\hat{g}\left(e^{j\theta}\right)\right\}}{\partial\theta} \leq \frac{m}{2} \qquad (13)$$

for $\theta \in (0,\pi)$. Substituting Eq. (11) with $n\left(e^{j\theta}\right) := n_r(\theta)+jn_i(\theta)$ into inequality (13), we obtain inequality (12). □

Let us remember that $n(z)$ is the numerator polynomial of the transfer function of map (1), at the fixed point $\boldsymbol{x}^*$, from the input $u_i(n)$ to the output $y_i(n)$. Note that the direction of $\hat{g}(z)$ depends only on $n(z)$ but not the connection parameters k and τ. This fact suggests that the direction cannot be changed by the connection parameters. Now all the Lemmas mentioned above allow us to solve our design problem.

Theorem 1. *Assume that maps (1) satisfy the following conditions: $\boldsymbol{A}$ does not have an odd number of real eigenvalues greater than 1; the maps are not scalar maps (i.e., $m \geq 2$); the polynomial $n(z)$ satisfies Lemma 3. If the connection parameters k and τ are designed such that both of $g(z,0)$ and $g(z,2)$ are stable, then steady state (3) is stable for any topology $\boldsymbol{E}$ and any number of maps N.*

Proof. It is obvious from the Lemmas; we omit this proof. □

Theorem 1 gives us the following procedure.

(Step 1) If $m \geq 2$ holds and $d(z) = \det\left[z\boldsymbol{I}_m - \boldsymbol{A}\right] = 0$ does not have odd number of real roots greater than 1, go to the next step, otherwise stop.

(Step 2) If $n(z) = \boldsymbol{c}\mathrm{adj}(z\boldsymbol{I}_m - \boldsymbol{A})\boldsymbol{b}$ satisfies Lemma 3, then go to the next step, otherwise stop.

(Step 3) Design k and τ such that both of

$$\begin{aligned} g(z,0) &= z^\tau d(z) + n(z)k\left(z^\tau - 1\right), \\ g(z,2) &= z^\tau d(z) + n(z)k\left(z^\tau + 1\right), \end{aligned} \qquad (14)$$

are stable.

5. Numerical Examples

This section will check our analytical results by some numerical examples. Let us consider the delayed logistic maps ($m = 2$) [7],

$$\boldsymbol{F}(\boldsymbol{x}) := \begin{bmatrix} x_{(2)} \\ px_{(2)}\left\{1 - x_{(1)}\right\} \end{bmatrix},\ \boldsymbol{b} := \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \boldsymbol{c} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T,$$

where p is the parameter. The fixed point is $\boldsymbol{x}^* = \left[(p-1)/p \quad (p-1)/p\right]^T$ and the Jacobi matrix $\boldsymbol{A}$ at fixed point $\boldsymbol{x}^*$ is

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 1-p & 1 \end{bmatrix}.$$

From $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$, we can obtain

$$d(z) = z^2 - z + p - 1,\ n(z) = z - 1. \qquad (15)$$

The parameter is fixed at $p = 2.1$ in accordance with our previous study [7].

Now we follow our procedure. For (Step 1) we estimate all the roots of $d(z) = 0$ with $m = 2$: $z = 0.5 \pm j\sqrt{3.4}/2$. As the odd number property does not hold, go to the next step. For (Step 2) we see that inequality (12) with $n_r(\theta) = \cos\theta - 1$ and $n_i(\theta) = \sin\theta$ is equivalent to $1/2 \leq 1$. This fact guarantees that $\hat{g}(z)$ is a convex direction, then go to the next step. For (Step 3) we set $k = 0.2$ and $\tau = 1$ such that $g(z,0)$ and $g(z,2)$ are stable.

In order to check that the designed k and τ are valid for any network topology, we prepare two typical networks consisting of six maps ($N = 6$): a ring network and a complete (i.e., all to all) network. Figures 1(a) and 1(b) show the time series data of $x_{(1)}$ for all the maps on the ring and the complete networks, respectively. All the maps without connection (i.e., $k \equiv 0$) runs for $n < 50$; they are connected at $n = 50$, then all the maps converge on the fixed point $\boldsymbol{x}^*$. These numerical results support our analytical results.

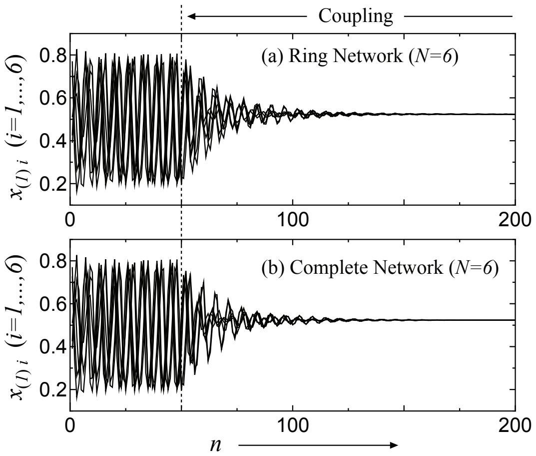

Figure 1: Time series data of of $x_{(1)}$ on two networks with $N = 6$: (a) ring network, (b) complete network.

6. Conclusions

This paper investigated amplitude death in high-dimensional map networks with connection delays. On the basis of the robust control theory, the stability of death was analyzed and the systematic procedure for designing the connection parameters was proposed. The designed parameters do not depend on the network topology and the number of maps. Our analytical results were confirmed by the delayed logistic map networks.

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A. Convex direction

Consider a family of m-th degree polynomials,

$$\begin{aligned} \delta(z) &= \{\mu\delta_1(z) + (1-\mu)\delta_2(z) : \mu \in [0,1]\}, \\ &= \left\{\delta_2(z) + \mu\hat{\delta}(z) : \mu \in [0,1]\right\}, \end{aligned} \tag{16}$$

where $\hat{\delta}(z) := \delta_1(z) - \delta_2(z)$ is a $(m-1)$-th degree polynomial. The definition of the convex direction is as follows.

Definition 1 ([14]). *A $(m-1)$-th degree polynomial $\hat{\delta}(z)$ is said to be a convex direction if, for all the stable m-th degree polynomials $\delta_2(z)$, m-th degree polynomial $\delta_2(z) + \hat{\delta}(z)$ is stable, then the family of m-th degree polynomials $\delta(z)$ is stable.*

This definition suggests that if m-th degree polynomials $\delta_{1,2}(z)$ are stable and $(m-1)$-th degree polynomial $\hat{\delta}(z)$ is a convex direction, then the family of m-th degree polynomials $\delta(z)$ is stable. The following theorem provides us a simple procedure to check if $\hat{\delta}(z)$ is a convex direction or not.

Theorem 2 ([14]). *A $(m-1)$-th degree polynomial $\hat{\delta}(z)$ is a convex direction if and only if the following inequality holds:*

$$\frac{\partial \arg\left\{\hat{\delta}\left(e^{j\theta}\right)\right\}}{\partial\theta} \leq \frac{m}{2} + \left|\frac{\sin\left(2\arg\left\{\hat{\delta}\left(e^{j\theta}\right)\right\} - m\theta\right)}{2\sin\theta}\right| \tag{17}$$

for $\theta \in \left\{\phi \in (0,\pi); \hat{\delta}\left(e^{j\phi}\right) \neq 0\right\}$.

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