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Overcoming the odd number property of amplitude death by an unstable dynamic connection

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Abstract—This paper investigates amplitude death induced by an unstable dynamic connection. This connection can be considered as an extension of the dynamic feedback controller based on an unstable low pass filter. We prove that the unstable dynamic connection can induce amplitude death at the steady state with the well-known odd number property. The analytical results are verified by numerical simulations.

1. Introduction

The collective dynamics in coupled nonlinear oscillators have been of great interest [1]. A stabilization of unstable steady state in diffusively coupled oscillators is known as amplitude death [2]. This phenomenon was discovered first in *non-identical* oscillators coupled by a static connection [3]. After that, it has been reported that a delayed connection [4], a dynamic connection [5], and a conjugate connection [6] can induce amplitude death even in coupled *identical* oscillators. These three connections have been experimentally verified by electronic circuits [7–9].

Amplitude death has been expected to suppress undesired oscillations in engineering systems such as DC micro grid [10] and coupled laser systems [11]. Thus, from engineering point of view, it is desirable to provide a simple design procedure of the connection parameters for inducing amplitude death. However, the delayed connection and the conjugate connection have the following disadvantages to practical use: the delayed connection is difficult to analyze, since its characteristic equation includes a delay term; the conjugate connection may cause oscillation death which is totally different from amplitude death [12]. Therefore, our previous study focused on the dynamic connection and proposed a topology-independent design procedure of the connection parameters [13]. However, the dynamic connection has a problem that it cannot stabilize the steady state if the Jacobian matrix of the oscillator at the steady state has an odd number of real positive eigenvalues, known as the odd-number property [13].

The odd number property was also reported for a single oscillator controlled by the dynamic feedback [14, 15]. To overcome this problem, these reports proposed a dynamic controller based on an unstable low pass filter; thus, the odd number property for a single oscillator has been already solved. On the other hand, the dynamic connection, an extension of the dynamic feedback to the coupled oscillators, has still the problem of the odd number property.

This paper proposes an unstable dynamic connection to overcome the odd number property in the coupled oscillators. This connection is an extension of the dynamic controller based on unstable low pass filter. It is analytically shown that the unstable dynamic connection can stabilize the steady state with the odd number property. These analytical results are verified by numerical simulations.

2. Unstable dynamic connection

2.1. A pair of oscillators

Let us consider two m-dimensional identical oscillators α and β,

$$\begin{cases} \dot{\boldsymbol{x}}_{\alpha,\beta} = \boldsymbol{F}(\boldsymbol{x}_{\alpha,\beta}) + \boldsymbol{b}u_{\alpha,\beta} \\ y_{\alpha,\beta} = \boldsymbol{c}\boldsymbol{x}_{\alpha,\beta} \end{cases}, \tag{1}$$

where $\boldsymbol{x}_{\alpha,\beta} \in \mathbf{R}^m$ is the state variable of each oscillator. $y_{\alpha,\beta} \in \mathbf{R}$ and $u_{\alpha,\beta} \in \mathbf{R}$ are the output signal and the input signal, respectively. $\boldsymbol{F}(\boldsymbol{x}) : \mathbf{R}^m \to \mathbf{R}^m$ denotes the nonlinear function. $\boldsymbol{b} \in \mathbf{R}^{m\times 1}$ and $\boldsymbol{c} \in \mathbf{R}^{1\times m}$ are the input and output vectors, respectively. Oscillators α and β are coupled by the following dynamic connection:

$$\dot{w} = \gamma(y_\alpha + y_\beta - 2w), \tag{2}$$

$$u_{\alpha,\beta} = k(w - y_{\alpha,\beta}), \tag{3}$$

where $w \in \mathbf{R}$ is the additional variable and $k > 0$ is the coupling strength. The connection parameter $\gamma \in \mathbf{R}$ represents the stability of the dynamic connection itself (i.e., Eq. (2) with $y_\alpha = y_\beta = 0$): for $\gamma \geq 0$, the dynamic connection is stable; for $\gamma < 0$, it is unstable.

Even though the previous studies [5,13] consider only for $\gamma \geq 0$, this report employs $\gamma < 0$ (i.e., the unstable dynamic connection). Notice that this connection is an extension of the dynamic feedback controller based on an unstable low-pass filter [14, 15]. Furthermore, for $\gamma < 0$, the variable w diverges before coupling the oscillators. We will introduce a way to avoid this divergence in Sec. 3.

Oscillators (1) with coupling (2) and (3) have the steady state,

$$\left[\boldsymbol{x}_\alpha^T \ \boldsymbol{x}_\beta^T \ w\right]^T = \left[\boldsymbol{x}^{*T} \ \boldsymbol{x}^{*T} \ w^*\right]^T, \tag{4}$$

where $\boldsymbol{x}^*$ is an unstable steady state of the nonlinear function $\boldsymbol{F}(\boldsymbol{x})$ (i.e., $\boldsymbol{F}(\boldsymbol{x}^*) = 0$) and $w^* := \boldsymbol{c}\boldsymbol{x}^*$. The linearized Eq. (1) with (2) and (3) around steady state (4) is given by

$$\begin{bmatrix} \Delta\dot{\boldsymbol{x}}_\alpha \\ \Delta\dot{\boldsymbol{x}}_\beta \\ \Delta\dot{w} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} - k\boldsymbol{bc} & \boldsymbol{0} & k\boldsymbol{b} \\ \boldsymbol{0} & \boldsymbol{A} - k\boldsymbol{bc} & k\boldsymbol{b} \\ \gamma\boldsymbol{c} & \gamma\boldsymbol{c} & -2\gamma \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{x}_\alpha \\ \Delta\boldsymbol{x}_\beta \\ \Delta w \end{bmatrix}, \tag{5}$$

where $\Delta\boldsymbol{x}_{\alpha,\beta} := \boldsymbol{x}_{\alpha,\beta} - \boldsymbol{x}^*$ and $\Delta w := w - w^*$ are the perturbation from the steady state (4). $\boldsymbol{A} := \{\partial\boldsymbol{F}(\boldsymbol{x})/\partial\boldsymbol{x}\}_{\boldsymbol{x}=\boldsymbol{x}^*}$ is the Jacobian matrix.

The stability of the linear system (5) is governed by the characteristic equation,

$$G(s) := g_1(s)g_2(s) = 0, \tag{6}$$

where

$$\begin{aligned} g_1(s) &:= \det\left[s\boldsymbol{I}_m - \boldsymbol{A} + k\boldsymbol{bc}\right], \\ g_2(s) &:= (s + 2\gamma) \det\left[s\boldsymbol{I}_m - \boldsymbol{A} + k\left(1 - \frac{2\gamma}{s + 2\gamma}\right)\boldsymbol{bc}\right]. \end{aligned} \tag{7}$$

The steady state (4) is stable if and only if all the roots of $g_1(s) = 0$ and $g_2(s) = 0$ stay on the open left-half complex plane.

2.2. Odd number property

This subsection reviews the odd number property of the conventional dynamic connection [5,13]. Moreover, it is shown that the unstable dynamic connection can overcome the odd-number property.

Consider the conventional dynamic connection ($\gamma \geq 0$). For real positive s, we have

$$\lim_{s\to\infty} g_2(s) = +\infty, \tag{8}$$

and

$$\begin{aligned} g_2(0) &= 2\gamma\det\left[-\boldsymbol{A}\right] \\ &= 2\gamma\prod_{q=1}^{m}(-\sigma_q), \end{aligned} \tag{9}$$

where σ_q ($q = 1, \ldots, m$) are the eigenvalues of $\boldsymbol{A}$. Assume that $\boldsymbol{A}$ has an odd number of real positive eigenvalues. Then, we have $g_2(0) < 0$. From Eq. (8) and $g_2(0) < 0$, the equation $g_2(s) = 0$ has at least one real positive root on the real axis. Therefore, amplitude death never occurs for any k and $\gamma > 0$ if $\boldsymbol{A}$ has an odd number of real positive eigenvalues. This limitation is called the odd-number property.

Now, we consider the unstable dynamic connection ($\gamma < 0$). Remark that Eq. (8) is still held even for $\gamma < 0$. On the other hand, if $\boldsymbol{A}$ has an odd number of real positive eigenvalues, then we obtain $g_2(0) > 0$. From Eq. (8) and $g_2(0) > 0$, we cannot guarantee whether $g_2(s) = 0$ has real positive roots or not. In other words, the unstable dynamic connection can overcome the odd-number property.

3. Numerical example

Consider a pair of Lorenz systems, which is given by Eq. (1) with

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{bmatrix} p(x^{(2)} - x^{(1)}) \\ -x^{(1)}x^{(3)} + rx^{(1)} - x^{(2)} \\ x^{(1)}x^{(2)} - bx^{(3)} \end{bmatrix}. \tag{10}$$

The parameters are fixed at the well-known values,

$$p = 10, \quad r = 28, \quad b = 8/3, \tag{11}$$

where the individual Lorenz system behaves chaotically.

The Lorenz system (10) with the parameters (11) has three unstable fixed points $\boldsymbol{x}_\pm^* := [\pm(br - b)^{1/2}, \pm(br - b)^{1/2}, r - 1]^T$ and $\boldsymbol{x}_0^* := [0, 0, 0]^T$. We see that the Jacobian matrix $\boldsymbol{A}$ at $\boldsymbol{x}_0^*$ has an odd number of real positive eigenvalues (odd number property). This report focuses on the stability of $\boldsymbol{x}_0^*$.

The input and output vectors are set to

$$\boldsymbol{b} = [0\ 1\ 0\,]^T, \quad \boldsymbol{c} = [0\ 1\ 0\,]. \tag{12}$$

The Jacobian matrix at $\boldsymbol{x}_0^*$ is given by

$$\boldsymbol{A} = \begin{bmatrix} -p & p & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}, \tag{13}$$

whose eigenvalues are $\lambda_1 = -8/3$, $\lambda_2 = 11.8277$, and $\lambda_3 = -22.8277$.

By substituting Eqs. (12) and (13) into Eq. (7), we have the characteristic equations,

$$\begin{aligned} g_1(s) &= (s + b)\left\{s^2 + (1 + k + p)s + p(k - r + 1)\right\}, \\ g_2(s) &= (s + b)\left[s^3 + (2\gamma + 1 + p + k)s^2 + \right. \\ &\qquad \left. \{2\gamma + p(2\gamma + 1 - r + k)\}\, s - 2\gamma p(1 - r)\right]. \end{aligned}$$

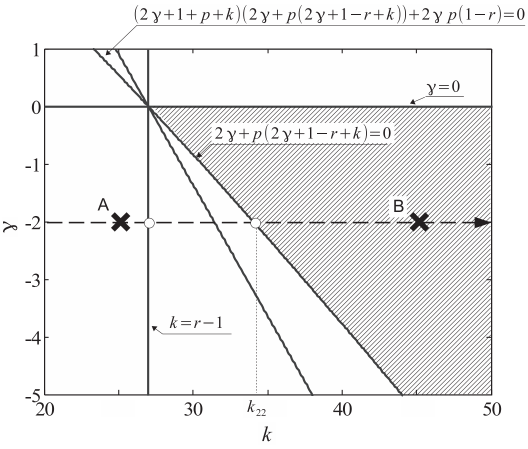


Figure 1: Stability region in (k, γ) space

The Routh-Hurwitz criteria shows that $g_1(s)$ and $g_2(s)$ are stable if and only if the following all inequalities are satisfied:

$$k - r + 1 > 0, \quad 2\gamma + 1 + p + k > 0,$$
$$2\gamma + p(2\gamma + 1 - r + k) > 0, \quad \gamma < 0,$$
$$(2\gamma + 1 + p + k)\{2\gamma + p(2\gamma + 1 - r + k)\} + 2\gamma p(1 - r) > 0. \quad (14)$$

The stability region on (k, γ) space is drawn in Fig. 1. The parameter sets (k, γ) in the shaded area satisfy all the inequalities (14). It should be noted that the stability region lies only for $\gamma < 0$ (i.e., the unstable dynamic connection).

Our analytical results are confirmed by numerical simulation. Since the dynamic connection (2) is unstable for $\gamma < 0$, the variable w diverges before coupling the oscillators. To avoid this divergence, Eq. (2) is modified as follows:

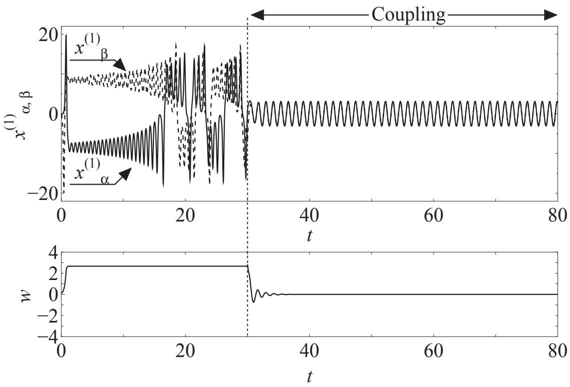
$$\dot{w} = \gamma(y_\alpha + y_\beta - 2\Phi(w)), \quad (15)$$

where $\Phi(x)$ is set to a piecewise linear function,

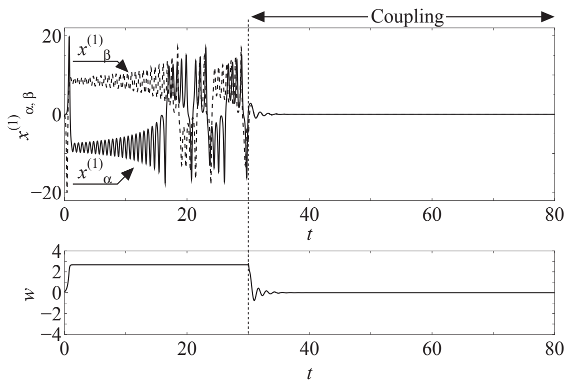
$$\Phi(x) := \begin{cases} -3x - 8 & \text{if } x < -2 \\ x & \text{if } -2 < x < 2 \\ -3x + 8 & \text{if } x > 2 \end{cases}. \quad (16)$$

Equation (15) without coupling ($y_\alpha = y_\beta = 0$) has two stable fixed points $w = 8/3$ and $w = -8/3$ and one unstable fixed point $w = 0$ which corresponds to the unstable dynamic connection. As a consequence, the variable w dose not diverge before coupling.

Figure 2 shows the time-series data of the variables $x^{(1)}_{\alpha,\beta}$ and w at point A:$(k, \gamma) = (25, -2)$ and point B:$(k, \gamma) = (45, -2)$ in Fig. 1. Two oscillators are coupled at $t = 30$. At point A as illustrated in Fig. 2(a),



(a) Point A:$(k, \gamma) = (25, -2)$

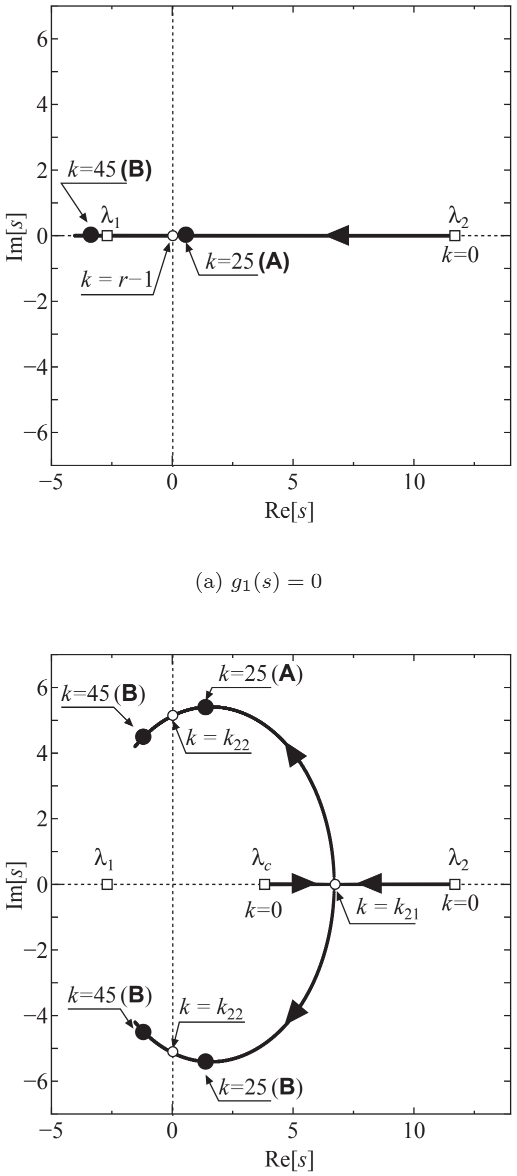


(b) Point B:$(k, \gamma) = (45, -2)$

Figure 2: Time-series data of $x^{(1)}_{\alpha,\beta}$ and w at points (a) A and (b) B in Fig. 1

after coupling, $x^{(1)}_{\alpha,\beta}$ and w still oscillate. In contrast, at point B as illustrated in Fig. 2(b), after coupling, they converge onto the fixed point $\boldsymbol{x}^*_0 = \boldsymbol{0}$ and $w^* = 0$ which satisfy the odd number property. Moreover, before coupling, the variable w dose not diverge but converges onto $w = 8/3$.

Here, to investigate the mechanism of stabilization, we derive the root locus of $g_1(s) = 0$ and $g_2(s) = 0$ as a function of k. The parameter γ is fixed at $\gamma = -2$ and k is varied from 0 to 50 (see the dashed arrow in Fig. 1). The root locus of $g_1(s) = 0$ is shown in Fig. 3(a). For $k = 0$, $g_1(s) = 0$ has one real positive root $s = \lambda_2$ corresponding to an eigenvalue of $\boldsymbol{A}$. With increasing k, this root moves to left and crosses the imaginary axis for $k = r - 1$. On the other hand, it is obvious from Eq. (7) that the real negative root $s = \lambda_1$ dose not move with increasing k. Figure 3(b) illustrates the root locus of $g_2(s) = 0$. For $k = 0$, $g_2(s) = 0$ have the two positive roots: $s = \lambda_2$ corresponding to the eigenvalue of $\boldsymbol{A}$, $s = \lambda_c$ corresponding



(a) $g_1(s) = 0$

(b) $g_2(s) = 0$

Figure 3: Root locus of $g_1(s) = 0$ and $g_2(s) = 0$ ($\gamma = -2$). k is varied from 0 to 50.

to the root of the unstable dynamic connection, where $\lambda_c := -2\gamma$. With increasing k, they close together and coalesce for $k = k_{21}$. After that, they turn to be the complex conjugate roots and move to left together. For $k = k_{22}$, they cross the imaginary axis from right to left. Therefore, Hopf bifurcation occurs for stabilization. Note that this stabilization process of $g_2(s) = 0$ is equivalent to that of a single oscillator with a dynamic feedback controller based on an unstable low pass filter [15].

4. Conclusion

This report investigated amplitude death induced by the unstable dynamic connection. We analytically indicated that the steady state with the odd number property can be stabilized by this connection. Furthermore, the obvious problem with the unstable connection has been solved by introducing the new stable fixed points in the connection. The analytical results are verified by numerical simulations.

Acknowledgments

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