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A Limit Cycle Control Method for Multi-Modal and 2-Dimensional Piecewise Affine Control Systems

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Abstract—This paper considers a limit cycle control problem of a multi-modal and 2-dimensional piecewise affine control system. First, we present a limit cycle synthesis problem and derive a solution of the problem. In addition, theoretical analysis on the rotational direction and the period of a limit cycle is shown. Next, the limit cycle control problem for piecewise affine control system is formulated. Then, we obtain matching conditions such that the piecewise affine control system with the state feedback law corresponds to the reference system which generates a desired limit cycle. Finally, in order to indicate the effectiveness of the new method, a numerical simulation is illustrated.

1. Introduction

Limit cycles are known to be quite important concept in various research fields and we can find limit cycles in real world. Researches on limit cycles have been eagerly done from both mathematical and engineering perspectives so far [1, 2, 3, 4]. Especially, some conditions for nonlinear systems that generate periodic solutions and some applications were shown in [1], and in [4], a synthesis method of hybrid systems whose solution trajectories converge to desired trajectories was proposed. In these studies, it is guaranteed that solution trajectories of the systems converges to a desired closed curve, and the existence of limit cycles was confirmed by numerical simulations. However, the mathematical guarantee of the existence of limit cycles was not shown. On the other hand, the authors proposed a synthesis method of multi-modal and 2-dimensional piecewise affine systems that generate desired limit cycles in [5, 6] and showed a mathematical proof of the existence and the uniqueness of a limit cycle for the proposed system. In addition, some theoretical analysis on the rotational direction and the period of a limit cycle is derived. In this study, we assume that the whole of a system can be designed. A method to generate a desired limit cycle for a given piecewise affine control system with tuning some parameters of the system is more useful for a wide variety of situations. However, such a control method have not been proposed so far. Hence, we consider a limit cycle control problem of multi-modal and 2-dimensional piecewise affine systems in this paper.

2. LIMIT CYCLE SYNTHESIS OF PIECEWISE AFFINE SYSTEMS

In this section, we explain a synthesis method of piecewise affine systems which generate desired limit cycles. Consider the 2-dimensional Euclidian space: $\mathbf{R}^2$, its coordinate: $x = [\ x_1 \ \ x_2\ ]^\top \in \mathbf{R}^2$, and the origin of $\mathbf{R}^2$: O. Let us set N ($N \geq 3$) points $P_i \neq O$ ($i = 1, \cdots, N$) in $\mathbf{R}^2$ and denote the vector from O to P_i by $p_i = [\ p_i^1 \ \ p_i^2\ ]^\top$. We also denote the angle between the half line OP_i and the x_1-axis by θ_i. Now, without loss of generality, we assume that the N points P_i ($i = 1, \cdots, N$) are located in the counterclockwise rotation from the x_1-axis, that is, $0 \leq \theta_1 < \theta_2 < \cdots < \theta_N$ holds. Next, we define the semi-infinite region D_i which is sandwiched by the half lines OP_i and OP_{i+1} and the line segment C_i joining P_i and P_{i+1}, where $P_{N+1} = P_1$. Set a polygon that is a union of C_i ($i = 1, \cdots, N$) as

$$C := \bigcup_{i=1}^{N} C_i. \tag{1}$$

Fig. 1 shows an example of a Polygonal Closed Curve for $N = 5$. We then consider the affine system defined in D_i:

$$\dot{x} = a_i + A_i x, \quad x \in D_i \tag{2}$$

where x is the state variable, and $a_i \in \mathbf{R}^2$, $A_i = \mathbf{R}^{2\times 2}$ are the affine term and the coefficient matrix, respectively. Hence, we treat the N-modal and 2-dimensional piecewise affine system (2).

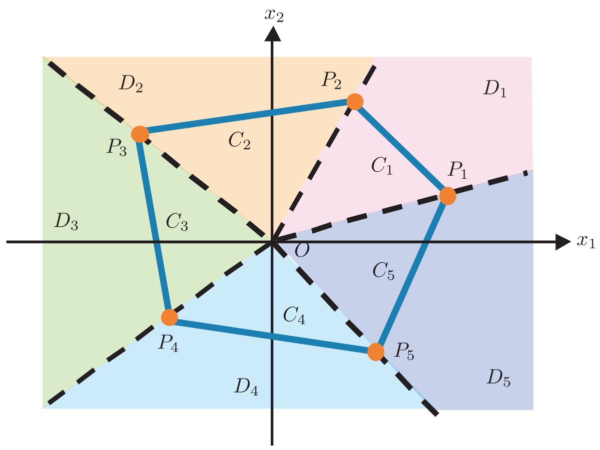


Fig. 1 : Example of Polygonal Closed Curve ($N = 5$)

The limit cycle synthesis problem for (2) is as follows.

Problem 1 : For the N-modal and 2-dimensional piecewise affine system (2), design $a_i,\ A_i\ \ (i = 1, \cdots, N)$ such that a given polygonal closed curve C (1) is a unique and stable limit cycle of the system. ∎

A solution of Problem 1 has been derived in the author's previous studies [5, 6], and then a_i and A_i of (2) can be designed as

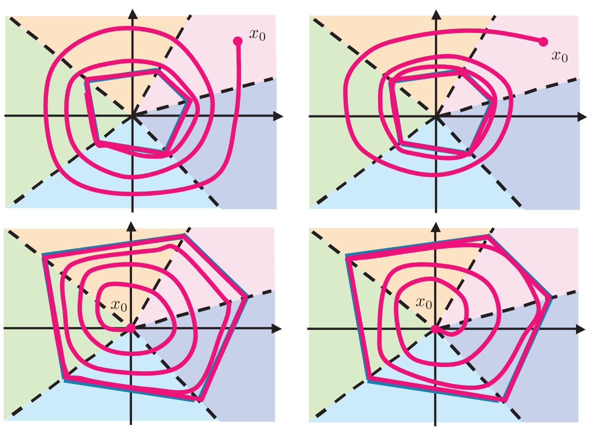
$$
\begin{aligned}
a_i &= \\
&\begin{bmatrix} -\lambda_i(p_i^2 - p_{i+1}^2)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1) - \omega_i(p_i^1 - p_{i+1}^1) \\ \lambda_i(p_i^1 - p_{i+1}^1)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1) - \omega_i(p_i^2 - p_{i+1}^2) \end{bmatrix}, \\
A_i &= \\
&\begin{bmatrix} -\lambda_i(p_i^2 - p_{i+1}^2)^2 & \lambda_i(p_i^2 - p_{i+1}^2)(p_i^1 - p_{i+1}^1) \\ \lambda_i(p_i^2 - p_{i+1}^2)(p_i^1 - p_{i+1}^1) & -\lambda_i(p_i^1 - p_{i+1}^1)^2 \end{bmatrix}.
\end{aligned} \tag{3}
$$

the main theorem on the existence of the limit cycle of the system (2) with (3) is as follow [5, 6].

Theorem 1 : For the N-modal and 2-dimensional piecewise affine system (2) with (3), assume that $\omega_i > 0\ \ (i = 1, \cdots, N)$ or $\omega_i < 0\ \ (i = 1, \cdots, N)$ holds. Then, the unique and stable limit cycle of the system (2) with (3) is equivalent to C. ∎

It is known that the system (2) with (3) has some important properties. Now, the definition of rotational directions of solution trajectories of the system (2) with (3) is given by the next.

Definition 1 [5, 6] : For limit cycle solution trajectories of the N-modal and 2-dimensional piecewise affine system (2) with (3), one that rotates in the clockwise direction in $\mathbf{R}^2$ is called *a limit cycle solution trajectory in the clockwise rotation*. On the contrary, one that rotates in the counterclockwise direction in $\mathbf{R}^2$ is called *a limit cycle solution trajectory in the counterclockwise rotation* (see Fig. 2). ∎



[a] Crockwise Rotation [b] Countercrockwise Rotation

Fig. 2 : Clockwise and Counterclockwise Rotations of Limit Cycle Solution Trajectories

The relationship between rotational directions of limit cycles and the parameters in (3) is shown in the following proposition.

Proposition 1 : For the N-modal and 2-dimensional piecewise affine system (2) with (3), its limit cycle solution trajectory moves in the counterclockwise rotation for $\omega_i > 0\ \ (i = 1, \cdots, N)$, and conversely it moves in the clockwise rotation for $\omega_i < 0\ \ (i = 1, \cdots, N)$. ∎

From Proposition 1, it is confirmed that the rotational directions of limit cycles do not depend on λ_i. Then, periods of limit cycles of the system (2) with (3) can be characterized by the next proposition.

Proposition 2 : When a limit cycle solution trajectory of the N-modal and 2-dimensional piecewise affine system (2) with (3) is sufficiently close to C, the period with which it rotates around C is given by

$$
T \approx \sum_{i=1}^{N} \frac{1}{|\omega_i|}. \tag{4}
$$

∎

From Proposition 2, we can also see that the period of a limit cycle solution trajectory of the system (2), (3) is not independent of λ_i. So, we can freely choose the value of λ_i.

3. LIMIT CYCLE CONTROL FOR PIECEWISE AFFINE SYSTEMS

3.1. Formulation of Limit Cycle Control

In this section, we consider a controller design problem on generation of limit cycles for given piecewise affine control systems. First, this sub-section gives the problem formulation. Consider the next piecewise affine control system defined in D_i:

$$
\dot{x} = a_i + A_i x + b_i u, \quad x \in D_i, \tag{5}
$$

where $u \in \mathbf{R}$ is the control input and $b_i \in \mathbf{R}^2$ is the coefficient vector for the control input. We next consider the state feedback law:

$$
u = k_i x + l_i, \quad x \in D_i, \tag{6}
$$

where $k_i \in \mathbf{R}^2$ and $l_i \in \mathbf{R}$. We assume that p_i, p_{i+1}, a_i, A_i are given parameters. Now, we formulate a problem on generating a desired limit cycle for the piecewise affine control system (2) and the state feedback law (6) as follows.

Problem 2 : For the N-modal and 2-dimensional piecewise affine control system (5) with the state feedback law (6), design $b_i,\ k_i,\ l_i, \omega_i, \lambda_i\ \ (i = 1, \cdots, N)$ such that a given polygonal closed curve C (1) is a unique and stable limit cycle of the closed-loop system. ∎

Throughout this paper, we call Problem 2 *a limit cycle control problem for piecewise affine control system*.

3.2. Matching Conditions for Limit Cycle Control Problem

This subsection derives a solution method of Problem 2 for the piecewise affine control system (5) with the state feedback law (6). To fulfill this, we shall utilize the limit cycle synthesis method obtained in Section 2. The results in Section 2 show that the unique and stable limit cycle of the system (2), (3) coincides with C. Hence, by tuning design parameters $b_i,\ k_i,\ l_i, \omega_i, \lambda_i$ $(i = 1, \cdots, N)$, we conform the closed-loop system (5), (6) to the system (2), (3). We here call the system (2), (3) *the reference system*. Use the following notations for the system (2), (3):

$$a_i = \begin{bmatrix} a_i^1 \\ a_i^2 \end{bmatrix},\ A_i = \begin{bmatrix} A_i^{11} & A_i^{12} \\ A_i^{21} & A_i^{22} \end{bmatrix}, \quad b_i = \begin{bmatrix} b_i^1 \\ b_i^2 \end{bmatrix},\ k_i = \begin{bmatrix} k_i^1 & k_i^2 \end{bmatrix}. \tag{7}$$

Conditions such that the closed-loop system (5), (6) is consistent with the reference system (2), (3) can be obtained by the following theorem.

Theorem 3 : The N-modal and 2-dimensional piecewise affine control system (5) with the state feedback control law (6) is equivalent to the reference system (2), (3) if and only if *the matching conditions*:

$$a_i^1 + b_i^1 l_i = -\lambda_i (p_i^2 - p_{i+1}^2)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1) - \omega_i (p_i^1 - p_{i+1}^1) \tag{8}$$

$$a_i^2 + b_i^2 l_i = \lambda_i (p_i^1 - p_{i+1}^1)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1) - \omega_i (p_i^2 - p_{i+1}^2) \tag{9}$$

$$A_i^{11} + b_i^1 k_i^1 = -\lambda_i (p_i^2 - p_{i+1}^2)^2 \tag{10}$$

$$A_i^{12} + b_i^1 k_i^2 = \lambda_i (p_i^2 - p_{i+1}^2)(p_i^1 - p_{i+1}^1) \tag{11}$$

$$A_i^{21} + b_i^2 k_i^1 = \lambda_i (p_i^2 - p_{i+1}^2)(p_i^1 - p_{i+1}^1) \tag{12}$$

$$A_i^{22} + b_i^2 k_i^2 = -\lambda_i (p_i^1 - p_{i+1}^1)^2 \tag{13}$$

hold.

(Proof) Substituting (6) into (5), we get the closed-loop system:

$$\begin{aligned} \dot{x} &= a_i + A_i x + b_i(k_i x + l_i) \\ &= a_i + b_i l_i + (A_i + b_i k_i) x \\ &= \begin{bmatrix} a_i^1 + b_i^1 l_i \\ a_i^2 + b_i^2 l_i \end{bmatrix} + \begin{bmatrix} A_i^{11} + b_i^1 k_i^1 & A_i^{12} + b_i^1 k_i^2 \\ A_i^{21} + b_i^2 k_i^1 & A_i^{22} + b_i^2 k_i^2 \end{bmatrix} x. \end{aligned} \tag{14}$$

Comparing the components of the reference system (2), (3) and (14), we obtain the matching conditions (8)–(13). ∎

The matching conditions (8)–(13) consists of 6 algebraic equations, and 7 unknown variables: $b_i^1, b_i^2, k_i^1, k_i^2, l_i, \omega_i, \lambda_i$. Hence, by solving them under the condition $\lambda_i > 0$, we can obtain these unknown variables, that is, a solution of Problem 2.

4. SIMULATIONS

This section illustrates a numerical example in order to confirm the effectiveness of the results derived in the previous sections. We now give data of the polygon with $N = 4$ as $P_1 = (1,\ 0)$, $P_2 = (0,\ 1)$, $P_3 = (-1,\ 0)$, $P_4 = (0,\ -1)$, which is shown in Fig. 3.

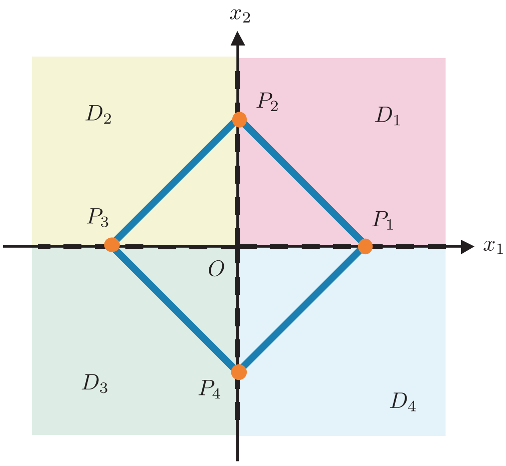


Fig. 3 : Polygonal Closed Curve of Example

The coefficients of the piecewise affine system are given by

$$\begin{aligned} a_1 &= \begin{bmatrix} -3 \\ 2 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}, \\ a_2 &= \begin{bmatrix} -1 \\ -3 \end{bmatrix},\ A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \\ a_3 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ A_3 = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}, \\ a_4 &= \begin{bmatrix} 9 \\ 1 \end{bmatrix},\ A_4 = \begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix}. \end{aligned} \tag{15}$$

By solving the matching conditions (8)–(13) for the problem setting, we can have design parameters as follows:

$$\begin{aligned} & b_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ k_1 = \begin{bmatrix} -1 & -2 \end{bmatrix}, \\ & l_1 = 1,\ \omega_1 = 3,\ \lambda_1 = 1, \\ & b_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix},\ k_2 = \begin{bmatrix} 1 & -2 \end{bmatrix}, \\ & l_2 = 2,\ \omega_2 = 1,\ \lambda_2 = 2, \\ & b_3 = \begin{bmatrix} -1 \\ 2 \end{bmatrix},\ k_3 = \begin{bmatrix} 1 & -2 \end{bmatrix}, \\ & l_3 = 2,\ \omega_3 = 2,\ \lambda_3 = 3, \\ & b_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ k_4 = \begin{bmatrix} 4 & 4 \end{bmatrix}, \\ & l_4 = -4,\ \omega_4 = 1,\ \lambda_4 = 4. \end{aligned} \tag{16}$$

Note that $\lambda_i > 0$ $(i = 1, 2, 3, 4)$ holds in (16). It can be confirmed that from Proposition 1, a limit cycle solution trajectory moves in the counterclockwise rotation since $\omega_i > 0$ $(i = 1, 2, 3, 4)$ holds. In addition, from Proposition

2 we can estimate the period of a limit cycle solution trajectory as

$$T \approx \sum_{i=1}^{4} \frac{1}{|\omega_i|} = \frac{25}{12}. \tag{17}$$

We set the initial state as $x_0 = [\,1,\ 1\,]^{\mathrm{T}}$ for the numerical simulation. The simulation results are illustrated in Figs. 4–6. Fig. 4 shows the solution trajectory on the $x_1 x_2$-plane. In Figs. 5 and 6, the time series of x_1 and x_2 are shown, respectively. From these simulation results, we can see that the solution trajectory that starts from x_0 behaves as a limit cycle for the desired polygonal closed curve C, and hence Theorem 1 holds. As we expected above, the solution trajectory moves in the counterclockwise rotation, and this result is coincident with Proposition 1. Moreover, the estimated period $T \approx 25/12$ is mostly agree about the simulation result from Figs. 5 and 6.

5. CONCLUSION

In this paper, we have considered a limit cycle control problem for a multi-modal and 2-dimensional piecewise control affine system. We have derive the matching conditions such that the piecewise control affine system with the state feedback law corresponds with the reference system which generates a unique and stable limit cycle. It has been confirmed by solving the matching conditions we can obtain the values of design parameters. A numerical simulations show the availability and the application potentiality of the proposed method.

Our future work includes applications of the proposed control method to real systems and extensions to multi-dimensional piecewise affine systems.

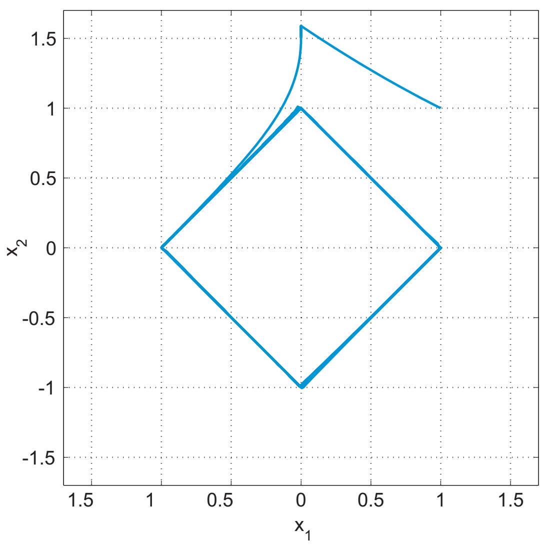


Fig. 4 : Solution Trajectory on $x_1 x_2$-Plane

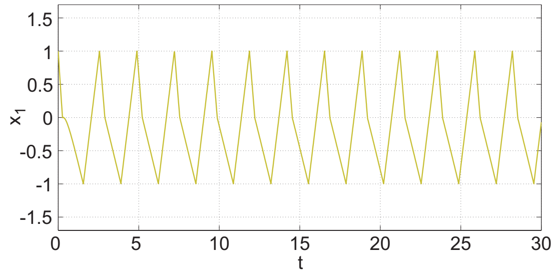


Fig. 5 : Time Series of x_1

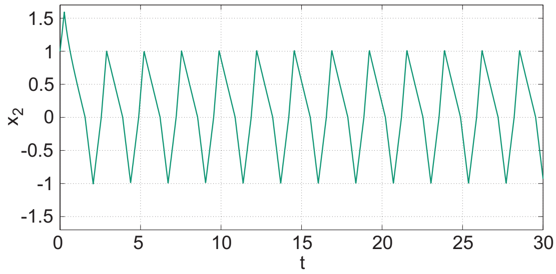


Fig. 6 : Time Series of x_2

Acknowledgments

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References

[1] D. N. Green, "Synthesis of Systems with Periodic Solutions Satisfying $\mathcal{V}(x) = 0$," *IEEE Trans. Circuits and Systems*, vol.31, no.4, pp.317–326, 1984

[2] S. N. Simic, K. H. Johansson, J. Lygeros and S. Sastry, "Hybrid Limit Cycles and Hybrid Poincare-Bendixson," *in Proc. of IFAC World Congress*, Barcelona, Spain, pp.86–89, 2002

[3] M. Adachi and T. Ushio, "Synthesis of Hybrid Systems with Limit Cycles Satisfying Piecewise Smooth Constant Equations," *IEICE Trans. Fundamentals*, vol.E87-A, No.4, pp.837–842, 2004

[4] A. Ohno, T. Ushio and M. Adachi, "Synthesis of Nonautonomous Systems with Specified Limit Cycles," *IEICE Trans. Fundamentals*, vol.E89-A, No.10, pp.2833–2836, 2006

[5] T. Kai and R. Masuda, "Limit Cycle Synthesis of Multi-Modal and 2-Dimensional Piecewise Affine Systems," *Mathematical and Computer Modelling*, Vol. 55, pp. 505–516, 2012

[6] T. Kai and R. Masuda, "A Limit Cycle Synthesis Method of Multi-Modal and 2-Dimensional Piecewise Affine Systems," *in Proc. of 50th IEEE Conference on Decision and Control and European Control Conference*, Orlando, USA, pp. 4759-4764, 2011