Non-Gaussian Order-Parameter Fluctuation in Complex Oscillator Network

Yasuhiro Tsubo

Department of Human and Computer Intelligence, Ritsumeikan University
1-1-1 Noji-Higashi, Kusatsu, Shiga 525-8577, JAPAN
Brain Science Institute, RIKEN, 2-1 Hirosawa, Wako, Saitama, 351-0198, JAPAN
Email: tsubo@ci.ritsumei.ac.jp

Abstract—Macroscopic signals or order parameters of complex oscillator network may reflect the structure of its network components. However, it remains unclear how to decipher network structure from macroscopic observables. Here, we develop an approach to relate the order parameter statistics of complex oscillator network to the structure of small-world networks.

1. Introduction

In complex systems consisting of many interconnected units, population activity of these units reflects the network topology and intrinsic properties of the elements. The intrinsic properties of each network component is able to be determined by isolation of a target unit from the complex network. On the other hand, it is more exacting to investigate the coupling structure between units in the complex network. For instance, in the neuronal complex network in the brain, many researchers have identified the network structure of complex network by simultaneously measuring from multiple elements and then extracting the strongly correlated pairs. However, obtaining activities of each element individually is often impossible in real recording systems, such as the neuronal network, and moreover, the strong "correlation" does not always mean the strong "coupling". So we desire to develop a novel strategy to investigate the network structure or topology of complex networks. We study how the macroscopic observables, coarse-grained or averaged signals of many surrounding element activities, depend on the network structure in complex noisy phase-oscillator systems.

In 1998, Bramwell et al. found that the power consumption in turbulent flow and the magnetization at critical point in magnetic XY-spin system show common statistical features: The probability distributions of these order-parameter fluctuations obey the same non-Gaussian distribution [1][2]. The phase order-parameter of coupled 2D-lattice oscillators with spatiotemporally independent weak noise and that of a 2D-coupled chaotic phase map model also obey similar non-Gaussian distribution [3]. Shape of the standardized non-Gaussian distribution is independent of the network system size, noise intensity (temperature), or diffusion factor if the system size is sufficiently large, while the average and the variance of the order parameter depend on these system parameters. On the other hand, the shape of the distribution function depends on the network topology such as the lattice dimension and the boundary condition [2][3]. For instance, the order parameter of the 2D-lattice system and that of the 3D-lattice system with the periodic boundary condition obey different non-Gaussian distribution. Here, we try to link the eigenvalues of Laplacian connection matrix with the non-Gaussian statistics of the phase order parameter in general phase oscillator networks.

2. Results

We introduce a coupled phase-oscillator model consisting of N phase-oscillator and their network topology is determined by Laplacian connection matrix L_{jk}. The time evolution of the phase of jth oscillator ϕ_j (mod 2π) is given by

$$\frac{d\phi_j(t)}{dt} = F(\phi_j(t)) + \sum_{k=1}^{N} W_{jk}(\phi_k(t) - \phi_j(t)) + c\xi_j(t),$$

where F is a nonlinear function determining the dynamics of jth phase oscillator ϕ_j and W_{jk} is a nonlinear interaction function between the jth and the $k$the oscillator. ξ_j and c represent spatiotemporally independent white Gaussian noise and its intensity, respectively. Suppose that all the oscillators are locked in phase when $c = 0$ and the noise intensity c is sufficiently small, we can approximate the nonlinear differential equations by a linearised form around the fixed point ($\phi = 0$) as

$$\frac{d\phi_j(t)}{dt} = \sum_{k=1}^{N} L_{jk}\phi_k(t) + c\xi_j(t).$$

We define a macroscopic phase order-parameter to this oscillator network as

$$M(t) = \frac{1}{N}\left|\sum_{j=1}^{N} e^{i\phi_j(t)}\right|.$$

The macroscopic observable $M(t)$ represents the degree of synchronization of N oscillators. If all the oscillators are perfectly synchronized, the value is unity. In the following subsection, we focus on the shape of distribution function of $M(t)$ for several complex networks, and then analytically derive the relationship between the distribution function of $M(t)$ and the network topology (structure of Laplacian connection matrix) in the last subsection.

2.1. Order-parameter distribution of 2D/3D-lattice network

As the first example of network topology, we show that the distribution of the phase order parameter $M(t)$ of 2D-square/triangle lattices with periodic boundary condition (Figure 1) [1]~[3]. As shown in Figure 1A, though the averages $\langle M \rangle$ and variances σ^2 of $M(t)$ depend on the shape of lattice (square or triangle) or the number of oscillators N, the standardized phase order-parameter $x = (M - \langle M \rangle)/\sigma$ obey the very skewed common non-Gaussian distribution (Figure 1B and C). Also in the 3D-lattice case, the standardized phase order-parameter x obeys another (less skewed) common non-Gaussian distribution independent of the network size (Figure 2).

Note that using different boundary conditions such as free boundary condition, the standardized phase order-parameter x obeys other common non-Gaussian distributions. Surprisingly, all the non-Gaussian distributions, including the 2D- and 3D-torus cases, are well fitted by a generalized Gumbel distribution family which has only one shape parameter κ [4]

$$P(x) = \frac{\nu}{\Gamma(\kappa)} \exp\left(\kappa(\nu x + \mu) - \exp(\nu x + \mu)\right),$$

where a scale parameters ν and a location parameter μ are determined so that the average of x is zero and the variance of x is unity.

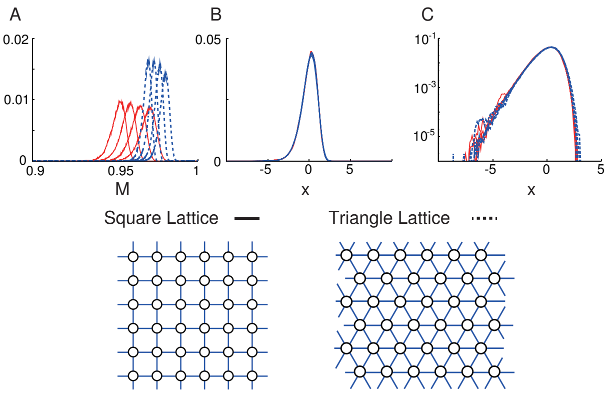

Figure 1: Order parameter distribution $P(M)$ and standardized order-parameter distribution $P(x)$ of 2D-torus phase-oscillator lattices ($N = 16^2$, 32^2, 64^2, 128^2, dashed lines:triangle lattices, solid lines:square lattice). A. Order parameter distribution ($P(M)$), B. Standardized distribution of A ($P(x)$), C. Semi-logarithmic plot of B ($\log P(x)$).

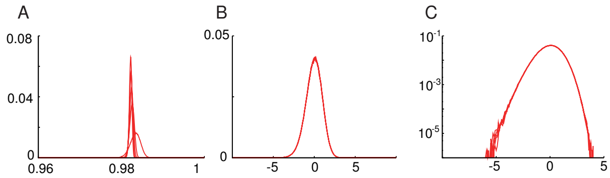

Figure 2: Order parameter distribution of 3D-torus phase-oscillator lattices ($N = 8^3$, 12^3, 16^3, 20^3, 24^3, 28^3). A. Order parameter distribution ($P(M)$), B. Standardized distribution of A ($P(x)$), C. Semi-logarithmic plot of B ($\log P(x)$).

2.2. Order-parameter distribution of Watts-Strogatz small-world network

The second example of network topologies is a small-world network introduced by Watts and Strogatz [4]. Watts-Strogatz network is generated by rewiring with probability p from a ring-like structure with nearest-k-element connection. The network is regular extended cycle if $p = 0$, whereas random network if $p = 1$. When p is set to an intermediate value (typically $0.01 \sim 0.1$), the network shows small-world properties (large-clustering and small-distance). Figure 3 A~C shows the p-dependence of standardized phase order-parameter distribution $P(x)$ in Watts-Strogatz network. Interestingly, in case of $p = 0.05$, $N = 2000$ (in the small-world regime), standardized phase order-parameter statistics obeys a non-Gaussian distribution which is very similar to that of lattice oscillator network shown in the previous subsection. In the case of Watts-Strogatz network, the shape of the phase order-parameter distribution $P(x)$ depends on the system size N (Figure 3D and F). Instead, if the "number of bypass connections" Npk is fixed (Figure 3E), the standardized phase order-parameter distributions fall on a common curve in the small-world regime (Figure 3F).

We find the standardized phase order-parameter obeys a non-Gaussian scale independent common distribution not only in 2D/3D lattices but also in Watts-Strogatz small world network.

2.3. Eigenvalues of Laplacian matrix

In order to find out the relationship between the shape of standardized phase order-parameter distribution and network topology, we derive the features of the eigenvalue set of Laplacian connection matrix from standardized phase order-parameter distribution distribution. We extended a method introduced in Bramwell et al. [2] for lattice networks to more general networks.

The time evolution of the phase oft the jth oscillator ϕ_j is described as

$$\phi_j(t) = \sum_{\alpha=1}^{N} \sum_{k=1}^{N} A_{jk}^{\alpha} c \int_0^t e^{-\lambda^\alpha s} \xi_k(t-s) ds$$

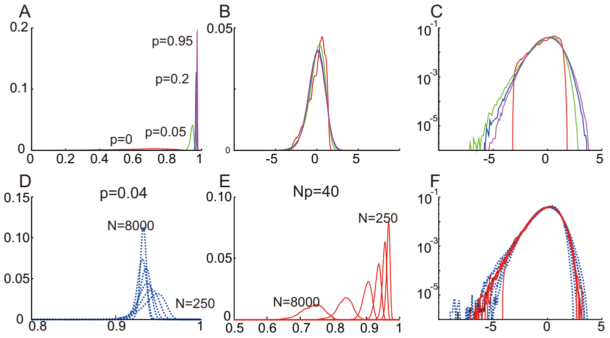

Figure 3: Order parameter distribution of Watts-Strogatz small world network ($k = 2$). A~C p-dependence ($p = 0, 0.05, 0.2, 0.95,\ N = 500$), B. Standardized distribution of A, C. Semi-logarithmic plot of B. D~F N-dependence ($N = 250,\ 500,\ 1000,\ 2000,\ 4000,\ 8000$). D. Rewiring probability is fixed ($p = 0.04$), E. Number of bypass connections is fixed ($Np = 40$), F. Semi-logarithmic plot of standardized distributions of D (dashed lines) and E (solid lines).

where $A^{\alpha}_{jk} \equiv \eta^{\alpha}_{j}\eta^{\alpha}_{k}$ and η^{α}_{j} is the eigenvector corresponding to the αth smallest eigenvalue λ^{α} of LCM L_{jk}. We introduce the phase difference $\psi_j(t) \equiv \phi_j(t) - \bar{\phi}(t)$, where $\bar{\phi}(t) \equiv \tan^{-1}\left(\sum_{j=1}^{N}\sin\phi_j(t)\Big/\sum_{j=i}^{N}\cos\phi_j(t)\right)$.

Using ψ_j, the average and correlation between two phases of oscillators in the steady state are calcurated as $\left\langle\psi_j\psi_k\right\rangle = \sum_{\alpha=2}^{N}\frac{c^2}{2\lambda^{\alpha}}A^{\alpha}_{jk} \equiv \epsilon J_{jk}$, $\left\langle\psi_j\right\rangle = 0$, where $\langle\cdot\rangle$ means ensemble average over the noise.

Therefore, the nth moment of the distribution of $M(t)$ is represented as

$$\langle M^n\rangle = \frac{1}{(2N)^n}\sum_{\{j\}=1}^{N}\exp\left(-\frac{\epsilon}{2}\sum_{\mu=1}^{n}J_{j_\mu j_\mu}\right) \times \operatorname{Tr}\exp\left(-\epsilon\sum_{\mu=1}^{n-1}\sum_{\nu=\mu+1}^{n}\sigma_\mu\sigma_\nu J_{j_\mu j_\nu}\right),$$

where $\sum_{\{j\}=1}^{N} = \sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\cdots\sum_{j_N=1}^{N}$ and trace Tr means the summation over all configuration of the set $\{\sigma = \pm 1\}$. In the case that network topology is statistically homogeneous, the above equation can be simplified as

$$\frac{\langle M^n\rangle}{\langle M\rangle^n} = \frac{1}{(2N)^n}\sum_{\{j\}=1}^{N}\operatorname{Tr}\exp\left(-\epsilon\sum_{\mu=1}^{n}\sum_{\nu=\mu+1}^{n}\sigma_\mu\sigma_\nu J_{j_\mu j_\nu}\right).$$

By using an approximation of exponential function in this equation with a Taylor expansion neglecting the higher order term and then transforming M to its standardized variable x, we can obtain the steady state distribution of x as

$$P(x) = \int_{-\infty}^{\infty}\frac{dy}{2\pi}\exp\left(ixy + \sum_{k=2}^{\infty}\frac{\left(i\sqrt{2}y\right)^k}{2k}R(k)\right) \quad (1)$$

$$R(k) \equiv \frac{\sum_{\alpha=2}^{N}(\lambda^{\alpha})^{-k}}{\left(\sum_{\alpha=2}^{N}(\lambda^{\alpha})^{-2}\right)^{k/2}}. \quad (2)$$

For given non-Gaussian SPO distribution $P(x)$, we can derive the set of moments $\{R(k)\}$ by using the Equaton (1). Then we can obtain the eigenvalues λ^{α} by solving the Equation (2).

3. Summary

In this study, we showed that the shape of order-parameter distribution in noisy phase-oscillator network systems is closely related to the network topology such as the number of bypass connections in Watts-Strogatz network. Then, we developed an approach to link the shape of a non-Gaussian distribution of the standardized phase order parameter to the eigenvalues of Laplacian connection matrix of a noisy phase-oscillator network. Our method of extracting the network structure from the macroscopic observables may be applied to monitor of the state of the network structure in sensor network, traffic network, power-supply network and neuronal network.

References

[1] Bramwell, ST, Holdsworth, PCW, & Pinton, JF. "Universality of rare fluctuations in turbulence and critical phenomena." *Nature*, 396, pp.552–554, 1998.

[2] Bramwell, ST, et al. "Universal Fluctuations in Correlated Systems." *Physical Review Letters*, vol.84, pp.3744–3747, 2000.

[3] Watanabe, T, Tsubo, Y, & Fujisaka, H. "Universality of chaotic rare fluctuations in a locally coupled phase map model." *Physical Review E*, vol.65, 026213, 2002.

[4] Watts, DJ, & Strogatz, SH. "Collective dynamics of 'small-world' networks." *Nature*, vol.393, pp.409–410,1998.