2022 International Symposium on Nonlinear Theory and Its Applications,
NOLTA2022, Virtual, December 12-15, 2022

Learning Koopman Eigenfunctions and Invariant Subspaces from Data: Symmetric Subspace Decomposition

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Nonlinear dynamical systems are ubiquitous in the world surrounding us. Naturally, understanding and controlling such systems is an important goal of science and engineering. Over the past century, state-space and geometric viewpoints have been the main tools for analysis and control of nonlinear phenomena. However, due to advances in digital computation and data acquisition, we are facing new challenges regarding efficient data-driven methods to understand and control complex nonlinear dynamical systems for which traditional geometric viewpoints are not sufficient.

Koopman operator theory, although not new, has shown great potential to address computational challenges raised by large data sets. The main reason is, independently of the type of nonlinearity in the system, the Koopman operator is linear and its eigenfunctions evolve linearly on the trajectories of the system. This linearity can lead to highly efficient numerical methods for system identification and control. However, it should be noted that the Koopman operator is generally infinite dimensional; hence its direct use on digital computers might require infinite resources. To tackle this issue, one can focus the attention on finite-dimensional spaces of functions and analyze the effect of the Koopman operator on them. However, the quality of such subspaces directly impacts the accuracy of the Koopman approximations. The approximation is exact if the finite-dimensional space is invariant under the effect of the operator. Hence, it is of utmost importance to identify Koopman-invariant subspaces from data.

Given that Koopman eigenfunctions naturally span invariant subspaces, our first task is to design numerical methods to find Koopman eigenfunctions. Moreover, since in practical settings only finitely many data points are available, we formulate our problem as a search in an *arbitrary* finite-dimensional space of functions spanned by a dictionary. Our primary goal is to find all Koopman eigenfunctions in this space.

A necessary condition for a function in the space to be a Koopman eigenfunction is to be captured by the eigendecomposition of the solution of the well-known Extended Dynamic Mode Decomposition (EDMD) algorithm. We build on this observation and provide a necessary and sufficient condition for a function to evolve linearly on data based on applying EDMD forward and backward in time and comparing the eigendecompositions. Furthermore, we prove that given additional reasonable conditions on the data sampling, the captured functions are all Koopman eigenfunctions in the original space almost surely. The identified eigenfunctions span a Koopman-invariant subspace. However, this subspace might not be maximal since this method does not capture generalized Koopman eigenfunctions.

To capture the maximal Koopman-invariant subspace in the original space, we provide an algorithmic procedure termed Symmetric Subspace Decomposition (SSD) that iteratively prunes the original space by reducing its dimension such that after finite iterations, it captures all linear evolutions in the data set. Under reasonable conditions on the data sampling, we prove that the solution of the SSD algorithm is the maximal Koopman-invariant subspace of the original space of functions almost surely. Moreover, given that in some applications, the data becomes available in a streaming fashion, we provide an extension of our algorithm termed Streaming Symmetric Subspace Decomposition (SSSD) that at each iteration, updates the previous solution based on the availability of new data. SSSD requires a small fixed memory independently of the size of the data set. As a result, SSSD can also be useful for invariant subspace calculations using large data sets.

Finally, in some practical cases, the maximal Koopman-invariant subspace of the original space might not contain sufficient eigenfunctions to describe the behavior of the system. To tackle this issue, we provide the Approximated-SSD algorithm to allow for capturing more information about the system's behavior while tuning the accuracy of the approximation.

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This work was supported by ONR Award N00014-18-1-2828.

A detailed description of the material discussed in this abstract can be found in [1].

References

[1] M. Haseli and J. Cortés, “Learning Koopman eigenfunctions and invariant subspaces from data: Symmetric Subspace Decomposition,” *IEEE Transactions on Automatic Control*, vol. 67, no. 7, pp. 3442–3457, 2022.