A Method to Obtain Particular Solutions using Modified Fourier Motzkin Method in P/T Petri Nets

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Abstract— P/T Petri nets are one kind of basic and useful models for discrete event systems. And reachability problem is one of the most important behavioral properties of Petri nets in determining its behavior. To consider the reachability problem, from an initial state called an initial marking, M_0 to a destination state called a destination marking M_d are the fundamental problems of Petri nets. There are some methods to solve such reachability problems, one of such methods is to use the coverability(reachability) tree, but this method requires a huge amount of calculation in general, so the other method to use matrix equations and reduction techniques has the advantage, because the method can utilize the algebraic equation properties of Petri nets. In this paper, we propose a modified algorithm of the Fourier-Motzkin method which is well known as a solution of the state equation for the reachability problem. The solutions which could not be found by a conventional algorithm can be obtained by using the modified one.

1. Introduction

A Petri net is a particular kind of directed graph, together with an initial state called the initial markings, M_0. The underlying graph of a Petri net is a directed, weighted, bipartite graph consisting of two kinds of nodes, called places and transitions, where arcs are either from a place to a transition or from a transition to a place. In graphical representation, places are drawn as circles, transitions as bars, and arks are labeled with their weights. The behavior of systems can be described in terms of system, a state or marking in a Petri nets is changed according to the transition firing rules. Such Petri nets are effectively used for modeling, analyzing, and verifying many discrete event systems[1].

In this paper, we are concerned with structural analysis based on the linear algebra techniques and the state equation $Ax = b := M_d - M_0$, where M_0 and M_d are initial and destination marking vectors, respectively. All generators for T-invariants and all minimal inhomogeneous(i.e., particular) solutions are needed for discussing the feasibility of a group of firing count vectors, x, for the fixed $b := M_d - M_0$ [2],[3], where any firing count vector is expanded by means of T-invariant generators and particular solutions [3]. We also consider to modify the algorithm of Fourier-Motzkin Method to find the solutions which could not be obtained by using conventional Fourier-Motzkin Method.

In section 2, preliminaries are given, and modified algorithm of Fourier-Motzkin method are described using an example for finding the solutions which could not be obtained by conventional method using the modified algorithm in section 3. And section 4 is the conclusion of this paper.

2. Preliminaries

2.1. State Equation

If the destination marking M_d was assumed to be reachable from initial marking M_0 through the firing sequence as $\{t_1, t_2, \cdots, t_d\}$, the state equation can be expressed as

$$M_d = M_0 + A\sum_{k=1}^{d} t_k \tag{1}$$

and eq.(1) can be described like as eq.(2) when $A \in Z^{m\times n}$, $b = M_d - M_0 \in Z^{m\times 1}$, $x = \sum_{k=1}^{d} t_k \in Z_+^{n\times 1}$

$$Ax = b. \tag{2}$$

Then we can obtain the firing count vector x to solve the solutions of eq.(2), from initial marking $M_0 \in Z_+^{m\times 1}$ to destination marking $M_d \in Z_+^{m\times 1}$.

2.2. Fourier-Motzkin Method

The Fourier-Motzkin method is to obtain the set of all elementary vector solutions as the nonnegative integer solutions of $Ax = 0^{m\times 1}$. And the algorithm of the Fourier-Motzkin method is as follows [4][5].

<Algorithm of Fourier-Motzkin method>
Input: Incidence matrix $A \in Z^{m\times n}$, m, and n.
Output: The set of T-invariants including all minimal support T-invariants.

Initialization: The matrix B is constructed by adjoining the identity matrix $E^{n\times n}$ to the bottom of the incidence matrix $A \in Z^{m\times n}$, with $B = [A^T, E]^T \in Z^{(m+n)\times n}$.
Step0: $i = 1$.
Step1: Select the i-th row of B. If the i-th row has no nonzero element, then $i = i + 1$ and go to Step2. If the i-th row has at least one nonzero element, then go to Step3.
Step2: If $i \leqq m$ is satisfied, go to Step1, otherwise go to Step4.
Step3: Add to the matrix B all the columns which are linear combinations of pairs of columns of B and which annul the i-th row of B. And eliminate from B the columns in which the i-th element is nonzero. Now, let us call the new matrix as B again. Then set $i = i + 1$ and go to Step2.
Step4: Each column of the submatrix which is obtained by deleting the rows of the first to the m-th from B is a minimal nonnegative integer solution for $Ax = 0^{m\times 1}$.

But, this method can be applied to $Ax = 0$, and this means that obtained solutions are T-invariants. So, to obtain the particular solutions(firing count vectors), we need to make such changes to the eq.(2) considering the augmented incidence matrix as follows:

$$\widetilde{A} = [\, A \quad -b \,] \in Z^{m\times(n+1)}. \tag{3}$$

then eq.(2) would be expressed by eq.(3) and augmented $\widetilde{x} \in Z^{n+1}$,

$$\widetilde{A}\widetilde{x} = 0. \tag{4}$$

Then, eq.(4) can be applied to the former algorithm.

3. Finding Particular Solutions which could not be Obtained by using The Conventional Algorithm

3.1. Finding Particular Solutions by using The Conventional Algorithm

Here, we would like to obtain particular solutions of an example Fig.1 by using the conventional algorithm of Fourier-Motzkin Method.

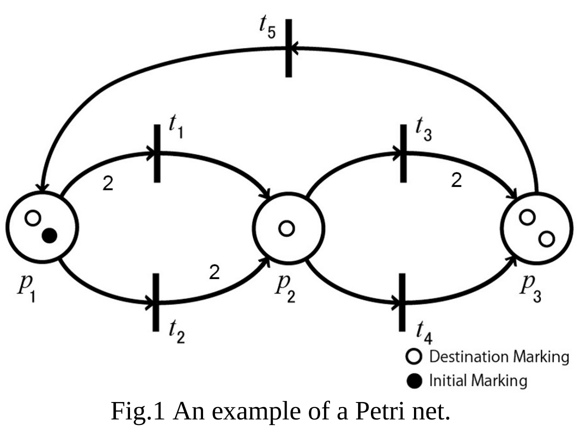

Fig.1 An example of a Petri net.

In this case, the incidence matrix of $A \in Z^{m\times n}$ is

$$A = \begin{bmatrix} -2 & -1 & 0 & 0 & 1 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 2 & 1 & -1 \end{bmatrix} \in Z^{3\times 5},$$

and the difference of marking $b \in Z^{m\times 1}$ from $M_0 \in Z_+^{m\times 1}$ to $M_d \in Z_+^{m\times 1}$ is

$$b = M_d - M_0 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \in Z^{3\times 1}.$$

Then the augmented matrix of A can be described as follows:

$$\widetilde{A} = \begin{bmatrix} -2 & -1 & 0 & 0 & 1 & 0 \\ 1 & 2 & -1 & -1 & 0 & -1 \\ 0 & 0 & 2 & 1 & -1 & -2 \end{bmatrix} \in Z^{3\times 6}$$

by eq.(3). And by the algorithm in §2.2, we can express the matrix B of Fig.1 using $B = [\, \widetilde{A} \;\; E \,] \in Z^{(m+n+1)\times(n+1)}$, as follows:

$$B = \left[\begin{array}{cccccc} -2 & -1 & 0 & 0 & 1 & 0 \\ 1 & 2 & -1 & -1 & 0 & -1 \\ 0 & 0 & 2 & 1 & -1 & -2 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]. \tag{5}$$

From matrix B of eq.(5), T-invariants and particular solutions are obtained by using the algorithm of the conventional Fourier-Motzkin method as follows:

$$\begin{aligned} &u_1 = (\,1\;0\;1\;0\;2\,)^T, \quad v_1 = (\,0\;3\;0\;5\;3\,)^T, \\ &u_2 = (\,1\;1\;0\;3\;3\,)^T, \end{aligned} \tag{6}$$

where $u_i \in U := \{u_i \in Z_+^{n\times 1}\}$; $Ax = b$ T-invariants and $v_j \in V := \{v_j \in Z_+^{n\times 1}\}$; $Ax = b$ particular solutions.

When we think about a particular solution of

$$v_j = (\,0\;2\;1\;2\;2\,)^T, \tag{7}$$

this solution is also the minimal vector like as eq.(6). Because not every element is smaller than the other solutions' elements. For example, the second element of eq.(7) **2** is smaller than v_1's second element of **3**, so eq.(7) can not be included by other obtained particular solution of v_1.

Hence, this means that we could not obtain all of the particular solutions by using the conventional algorithm of the Fourier-Motzkin Method.

3.2. Finding Particular Solutions by using a Modified Algorithm

All of the particular solutions could not be obtained by using conventional algorithm of the Fourier-Motzkin Method in §3.1. So, we guess the reason why, and this would be the algorithm of step 3 in §2.2 is not enough to find the all of combinations to make annul the i-th row of B.

In other words, annulment of the i-th row of B can be made by not only pairs of linear combinations, but more than 2 columns combinations. By using all of the combinations to make annul the i-th row of B, it would be able to find the new solutions which could not be found by using the conventional algorithm of Fourier-Motzkin Method.

Then the modified algorithm of the Fourier-Motzkin Method to improve this problem is as follows:

<Algorithm of Modified Fourier-Motzkin method>
<u>Input</u>: Incidence matrix $A \in Z^{m\times n}$, m, and n.
<u>Output</u>: All of minimal T-invariants.
<u>Initialization:</u> The matrix B is constructed by adjoining the identity matrix $E^{n\times n}$ to the bottom of the incidence matrix $A \in Z^{m\times n}$, where $B = [A^T, E]^T \in Z^{(m+n)\times n}$.
<u>Step0</u>: $i = 1$.
<u>Step1</u>: Select the i-th row of B. If the i-th row has no nonzero element, then $i = i + 1$ and go to Step2. If the i-th row has at least one nonzero element, then go to Step3.
<u>Step2</u>: If $i \leqq m$ is satisfied, go to Step1, otherwise go to Step8.
<u>Step3</u>: If the i-th row of B has at least one pair of positive and negative elements, go to Step4, otherwise go to Step7.
<u>Step4</u>: Aiming the i-th row of B(i.e., the old matrix), add directly the j-th column to the k-th column, where the (i, j) element is positive and the (i, k) element is negative. Apply the minimal vector criterion to the above new column vector and the column vectors each of which has the zero i-th element on the old matrix B. Adjoin all the remained columns after this criterion to the old matrix B. Then call this new matrix as B again. Then go to Step5.
<u>Step5</u>: If the i-th element of all the adjoined column vectors of the new matrix B is zero, go to Step7, otherwise go to Step6.
<u>Step6</u>: Repeat Step4 to the matrix B. However, the (j, k) pair should be always new. Then, go to Step5.
<u>Step7</u>: Delete, from B, all the columns each of which has nonzero element on the i-th row of B. Now, let us call the new matrix as B again. Then set $i = i + 1$ and go to Step2.
<u>Step8</u>: Each column of the submatrix which is obtained by deleting the rows of the first to the m-th from B is a minimal nonnegative integer solution.

So, let us try to obtain the solutions using this modified algorithm. Apply this algorithm to the eq.(5), then Matrix B is added to the columns which annul the first row of B made by positive and negative combination without weights, just as shown in eq.(8)

$$B = \left[\begin{array}{rrrrrr|rr} -2 & -1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & 2 & -1 & -1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 2 & 1 & -1 & -2 & -1 & -1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]. \tag{8}$$

In eq.(8), the 7th and 8th columns are added. And next, try to make more new columns by using added columns and other columns in combination again.

$$B = \left[\begin{array}{rrrrrrrr|r} -2 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 2 & -1 & -1 & 0 & -1 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 & -1 & -2 & -1 & -1 & -2 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]. \tag{9}$$

Then 9th column can be added as shown in eq.(9). But this column is not a minimal vector, because when considering about from 4th row to 9th row of 8th and 9th column, each element of the 9th column(added column) are not smaller than or equal to each element of the 8th column(already existing column). Then this 9th column is deleted from Matrix B.

Now there are no more combination columns to add to the Matrix B, so the operation to the first row is over. Then all the columns which have nonzero elements in the first row of B are deleted, and the rest of the columns will be the new Matrix B. And the same operation will be done to the next row of the new Matrix B.

$$B = \left[\begin{array}{rrrr|rrr} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 2 & 1 & 1 & 1 \\ 2 & 1 & -2 & -1 & 1 & 0 & -3 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]. \tag{10}$$

In the second row, the new columns from 5th to 7th by a combination of positive and negative from 1st column to

$$
B = \left[\begin{array}{ccccccc|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & -2 & -1 & 1 & 0 & -3 & 3 & 2 & -1 & 1 & -2 & -5 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2
\end{array}\right]. \tag{11}
$$

4th column are added as shown in eq.(10). The column operation in the second row can not apply anymore, so all the columns each of which have nonzero elements in the second row of B are deleted, and the rest of the columns will be the new Matrix B again, and the same operation will be applied to the next row. In the 3rd row, add the new

$$
B = \left[\begin{array}{cccccc|ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 2 & -1 & 1 & -2 & -5 & 2 & 1 & -2 & 0 & -3 & -1 & -4 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 1 & 1 & 0 & 0 & 0 & 3 & 2 & 2 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 3 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 1 & 0 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 2
\end{array}\right]. \tag{12}
$$

columns which are combinations of positive and negative. But in this case, the new combinations are increased by new combinations with added columns and existing columns. Then the algorithm can not be done to the end. Eq.(13)

$$
B = \begin{array}{c}
\begin{array}{ccccccccccccccccccc}
 & & & & & & & & & * & & & & & & & & ** &
\end{array} \\
\left[\begin{array}{ccccccccccccc|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
3 & 2 & -1 & 1 & -2 & -5 & 2 & 1 & -2 & 0 & -3 & -1 & -4 & 1 & 0 & -1 & -2 & 0 & \cdots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & \cdots \\
2 & 1 & 1 & 0 & 0 & 0 & 3 & 2 & 2 & 1 & 1 & 0 & 0 & 4 & 3 & 2 & 1 & 0 & \cdots \\
0 & 1 & 0 & 2 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 3 & 2 & 0 & 1 & 2 & 3 & 5 & \cdots \\
1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & \cdots \\
0 & 0 & 1 & 0 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 1 & \cdots
\end{array}\right]
\end{array}. \tag{13}
$$

is the calculation result to the middle, but on the 10th column(marked '*'), this solution is already obtained by using the conventional algorithm in eq.(6). And paying attention to the 18th column(marked '**'), this is the solution just shown in eq.(7). So, the new solution which could not be found by using the conventional algorithm could be obtained by using the modified algorithm of the Fourier-Motzkin Method, even though the algorithm could not be done.

4. Conclusions

To find T-invariants and particular solutions for state equation in P/T Petri nets can be obtained by using the Fourier-Motzkin Method. But some particular solutions could not be obtained by using the algorithm of the conventional Fourier-Motzkin Method. In this paper, some of the particular solutions which could not be obtained by using the conventional algorithm of the Fourier-Motzkin Method can be obtained by using a modified algorithm have been shown.

The algorithm is not completed because the calculation could not be done to the end. But a part of the unobtained particular solution could be found by using the modified algorithm.

In future studies, we would like to improve the algorithm to obtain all of the unobtainable solutions and also to deal bigger P/T Petri nets using the Fourier-Motzkin Method.

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