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A Solvable Condition of the Factorization Method for Polynomials Using the Inverse Z-transformation

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Abstract—In this paper, we discuss a property of the factorization method for polynomials using the inverse z-transformation. The method solves the factorization for polynomials by reducing the problem of two dimensional convolution to the problem of one dimensional convolution and solving several systems of linear equations. However, the method can't always obtain a factorization for all polynomials. Therefore, we show a necessary condition for solving by the method.

1. Introduction

Symbolic algebra system such as Mathematica or Maple is well known for solving numerical formulas. These systems usually treat strict solutions. However, in image processing, given images almost include observation error and can not treat strictly. So, we have proposed a restoring method for degraded images by using the inverse z-transformation [1]. The method cannot only restore degraded images by numerical calculation but also apply to factorize polynomials in the real field. However, in a rare case, a system of linear equations which generated in the algorithm can not be solve because of linear dependent. In this paper, we clarify the condition when the phenomenon happens in the factorization method.

In the following sections, we discuss the case of only two variables to avoid complicated notations but all results can extend even in the case of more than three variables.

2. Polynomials and Convolution

Let $G(x, y)$ be a given polynomial with two variables. Two polynomials $F(x, y)$ and $H(x, y)$ are its factors. Then, the given polynomial $G(x, y)$ is resolved to the product of these two polynomials as follows:

$$G(x, y) = H(x, y) \cdot F(x, y) \tag{1}$$

Let $G = g(i, j)$, $H = h(i, j)$ and $F = f(i, j)$ be matrices obtained by the inverse z-transformation of polynomials $G(x, y)$, $H(x, y)$ and $F(x, y)$, respectively. Here, the matrix G and the polynomial $G(x, y)$ satisfy

$$G(x, y) = \sum_{i=1}^{M} \sum_{j=1}^{N} g(i, j) x^{i-1} y^{j-1}, \tag{2}$$

where $M - 1$ and $N - 1$ are degrees of two variables x and y, respectively. Then, we get the relation among three matrices as follows:

$$G = H * F \tag{3}$$

where $*$ means convolution and we obtain

$$g(m, n) = \sum_{i=1} \sum_{j=1} h(m+1-i, n+1-j) f(i, j) \tag{4}$$

for $1 \leq m \leq M$ and $1 \leq n \leq N$.

Example 1 We try to obtain factors of a given polynomial

$$\begin{aligned} G(x, y) &= 6 + 10x + 4x^2 + 21y + 35xy + 14x^2 y \\ &\quad +9y^2 + 15xy^2 + 6x^2 y^2. \end{aligned} \tag{5}$$

By the inverse z-transformation of Eq. (5), we obtain a matrix

$$G = \begin{pmatrix} 6 & 21 & 9 \\ 10 & 35 & 15 \\ 4 & 14 & 6 \end{pmatrix}.$$

Put two matrices be

$$H = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 3 & 9 \\ 2 & 6 \end{pmatrix}.$$

Then, it holds

$$\begin{aligned} G &= \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} * \begin{pmatrix} 3 & 9 \\ 2 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 2\cdot 3 & 1\cdot 3+2\cdot 9 & 1\cdot 9 \\ 2\cdot 3+2\cdot 2 & 1\cdot 3+1\cdot 2+2\cdot 9+2\cdot 6 & 1\cdot 9+1\cdot 6 \\ 2\cdot 2 & 1\cdot 2+2\cdot 6 & 1\cdot 6 \end{pmatrix}. \end{aligned}$$

Hence, two polynomials obtained by the z-transformation of these two matrices

$$\begin{aligned} H(x, y) &= 2 + 2x + y + xy \\ F(x, y) &= 3 + 2x + 9y + 6xy \end{aligned}$$

satisfy the relation Eq. (1) and become two factors of the given polynomial $G(x, y)$.

In the next section, we shall explain a finding method of above two matrices H and F.

3. A Factorization Method for Polynomials

In the case of factorization for polynomials with one variable, it is known that every polynomials with n degree have n zero points in the imaginary number field [2]. Also, we can obtain those all zero points by using the Durand-Kerner-Aberth method (DKA method) that is a numerical calculation method [3]. By using the method, it is easy to obtain the factorization for polynomials with one variable in the mean of numerical calculations. However, in the case of more than two variables, it is impossible to obtain all factors of a given polynomial by the same method since it can't be always decomposed to several polynomials with one variable even in the imaginary field. Therefore, we proposed an improvement of the method for one variable to factorize polynomials with more than two variables. The proposed method makes factorization of a given polynomial by using the DKA method and solving some systems of linear equations.

The following algorithm is the proposed factorization method for polynomials with two variable to decompose two polynomials. However, by repeat of the method, we can obtain all irreducible polynomials of each given polynomial.

Factorization of polynomials with two variables

(i) Let $G(x, y)$ be a given polynomial with two variables and $g(m, n)$ be a $M \times N$ matrix obtained by the inverse z-transformation for the polynomial. Here, put

$$g(m) = \sum_{n=1}^{N} g(m, n) \quad (6)$$

for $m = 1, 2, \cdots, M$.

(ii) Let $G(x)$ be a polynomial obtained by the z-transformation of the matrix $g(m)$. Then, the polynomial $G(x)$ with one variable is factored to the product of irreducible factors in the real number field by using the DKA method.

(iii) Let suppose $K \times L$ to be the size of a finding matrix $h(m, n)$. If we can not select new size for the matrix, we stop. Otherwise, we try to find a polynomial $H(x)$ that is a product of some irreducible factors of $G(x)$ with $K - 1$ degrees. Then, from the coefficients of the expansion of the polynomial $H(x)$, we obtain sums p_k of each row of the finding matrix $h(m, n)$:

$$p_k = \sum_{n=1}^{L} h(k, n) \quad (7)$$

for $k = 1, 2, \cdots, K$. Here, to avoid multiple solutions by constant numbers, we add a condition of normalization such that $\sum p_k = 1$.

(iv) Let solve one dimensional factorization of the convolution of two matrices $h(1, n)$ and $f(1, n)$ by using $p_1 = \sum_{n=1}^{L} h(1, n)$ and the DKA method.

(v) For each $k = 2, 3, \cdots, M - 1$, we solve a system of $(N + 1)$ linear equations:

$$\begin{aligned} h(k, n) * f(k, n) &= g(k, n) \\ \sum_{n=1}^{L} h(k, n) &= p_k \end{aligned} \quad (8)$$

where $1 \leq n \leq N$. Then, we obtain all values of two matrices $h(m, n)$ and $f(m, n)$.

(vi) We verify that the convolution of obtained two matrices $h(m, n)$ and $f(m, n)$ is equal to $g(m, n)$. If the difference is less than a threshold, we obtain two finding polynomials $H(x, y)$ and $F(x, y)$ by applying the z-transformation to two matrices $h(m, n)$ and $f(m, n)$ and stop. Otherwise, return to the Step (iv) and try again with another selection of factors. If it is not sufficient, return to the Step (iii) and change the size $K \times L$ of the matrix $h(m, n)$.

Example 2 Let a given polynomial G(x,y) to be

$$\begin{aligned} G(x, y) = \; & 3 + 8x + 13y + 13x^2 + 25xy + 25y^2 \\ & +6x^3 + 31x^2y + 43xy^2 + 7y^3 \\ & +12x^3y + 49x^2y^2 + 8xy^3 \\ & +18x^3y^2 + 9x^2y^3. \end{aligned}$$

Then, by the inverse z-transformation, it is translated to a 4×4 matrix:

$$g(m, n) = \begin{pmatrix} 3 & 13 & 25 & 7 \\ 8 & 25 & 43 & 8 \\ 13 & 31 & 49 & 9 \\ 6 & 12 & 18 & 0 \end{pmatrix}.$$

Then, we get

$$\begin{aligned} g(1) &= 3 + 13 + 25 + 7 = 48 \\ g(2) &= 8 + 25 + 43 + 8 = 84 \\ g(3) &= 13 + 31 + 49 + 9 = 102 \\ g(4) &= 6 + 12 + 18 + 0 = 36. \end{aligned}$$

Next, in the real number field, we obtain only one irreducible factorization by the DKA method as follows:

$$\begin{aligned} G(x) &= 48 + 84x + 102x^2 + 36x^3 \\ &= 6(2 + x)(4 + 5x + 6x^2). \end{aligned}$$

Here, when we suppose the size of the matrix $h(m, n)$ is 2×2, the size of the other matrix $f(m, n)$ becomes 3×3. Therefore, it holds $H(x) = 2 + x$. From the normalization of coefficients, we obtain

$$\begin{aligned} p_1 &= \frac{2}{2 + 1} = \frac{2}{3} \\ p_2 &= \frac{1}{2 + 1} = \frac{1}{3}. \end{aligned}$$

Next, we solve one dimensional convolution $h(1,n) * f(1,n) = g(1,n)$. Since the factorization must satisfy each coefficient of factors of $G(1,y) = \sum_{j=1}^{4} g(1,j)y^{j-1}$ with a real number, we obtain only one factorization

$$\begin{aligned} G(1,y) &= 3 + 13y + 25y^2 + 7y^3 \\ &= (3+y)(1+4y+7y^2). \end{aligned}$$

Here, from $p_1 = \frac{2}{3}$, we have

$$h(1,n) = \frac{1}{6}\begin{pmatrix} 3 & 1 \end{pmatrix}, \quad f(1,n) = 6\begin{pmatrix} 1 & 4 & 7 \end{pmatrix}.$$

Next, we solve a system of 5 linear equations:

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & 6 & 0 \\ \frac{1}{6} & \frac{1}{2} & 0 & 24 & 6 \\ 0 & \frac{1}{6} & \frac{1}{2} & 42 & 24 \\ 0 & 0 & \frac{1}{6} & 0 & 42 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} f(2,1) \\ f(2,2) \\ f(2,3) \\ h(2,1) \\ h(2,2) \end{pmatrix} = \begin{pmatrix} 8 \\ 25 \\ 43 \\ 8 \\ \frac{1}{3} \end{pmatrix}$$

and we get

$$h(2,n) = \begin{pmatrix} \frac{1}{3} & 0 \end{pmatrix}, \quad f(2,n) = 6\begin{pmatrix} 2 & 5 & 8 \end{pmatrix}.$$

Finally, we solve a system of 3 linear equations gotten from the convolution for $1 \le l \le 3$

$$h(m,n) * f(m,n) = g(3,l),$$

we get

$$f(3,n) = 6\begin{pmatrix} 3 & 6 & 9 \end{pmatrix}.$$

Thus, we obtain two matrices

$$h(x,y) = \frac{1}{6}\begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}, \quad f(x,y) = 6\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}.$$

Finally, by the z-transformation, we get two irreducible factors as follows:

$$\begin{aligned} H(x,y) &= \frac{1}{6}(3 + 2x + y) \\ F(x,y) &= 6(1 + 2x + 4y + 3x^2 + 5xy + 7y^2 \\ &\quad +6x^2y + 8xy^2 + 9x^2y^2). \end{aligned}$$

4. A Solvable Condition for Simultaneous Linear Equations

The above algorithm for factorization of polynomials cannot always solve simultaneous linear equations in the Step (v). In a rare case, the simultaneous linear equations become linearly dependent. For example, let us consider the case:

$$\begin{pmatrix} 0.5 & 0 \\ h(2,1) & h(2,2) \end{pmatrix} * \begin{pmatrix} 2 & 0 \\ f(2,1) & f(2,2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, we obtain a system of linear equations as follows:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 0.5 & 0 \\ 0 & 2 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} h(2,1) \\ h(2,2) \\ f(2,1) \\ f(2,2) \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0 \\ 2 \\ 0 \end{pmatrix}.$$

In this time, we cannot solve this equation because all elements of the bottom line of the matrix are equal to 0 and the system obviously becomes linear dependent. However, the system has some solutions. For example,

$$H = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad F = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

is one of those solutions. Please remark that two matrices are the same form without a constant factor.

A solvable condition for simultaneous linear equations in the Step (v) is given by the following theorem.

Theorem 1 *In the Step* (v) *of the proposed algorithm, we can solve all systems of linear equations for* $k = 2, 3, \cdots, M-1$ *if the next inequality holds.*

$$\left\{\sum_{n=1}^{L} h(1,n)\right\} \cdot \begin{vmatrix} h(1,1) & \ddots & 0 & f(1,1) & \ddots & 0 \\ h(1,2) & \ddots & \vdots & f(1,2) & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & 0 \\ h(1,L-1) & \ddots & 0 & \vdots & \ddots & f(1,1) \\ h(1,L) & \ddots & h(1,1) & f(1,I) & \ddots & f(1,2) \\ 0 & \ddots & h(1,2) & 0 & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ddots & h(1,L) & 0 & \ddots & f(1,I) \end{vmatrix} \neq 0$$

where $I = N - L + 1$ *and the size of the determinant is* $(N+1) \times (N+1)$.

Proof: Every Systems of $N+1$ linear equations in the Step (v) always satisfy

$$\begin{pmatrix} h(1,1) & \ddots & 0 & f(1,1) & \ddots & 0 \\ h(1,2) & \ddots & \vdots & f(1,2) & \ddots & \vdots \\ \vdots & \ddots & 0 & \vdots & \ddots & 0 \\ h(1,L) & \ddots & h(m,1) & f(1,I) & \ddots & f(m,1) \\ 0 & \ddots & h(1,2) & 0 & \ddots & f(1,2) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ddots & h(1,L) & 0 & \ddots & f(1,I) \\ 0 & \ldots & 0 & 1 & \ldots & 1 \end{pmatrix}\begin{pmatrix} f(k,1) \\ f(k,2) \\ \vdots \\ \vdots \\ f(k,I) \\ h(k,1) \\ h(k,2) \\ \vdots \\ \vdots \\ h(k,L) \end{pmatrix}$$

$$
= \begin{pmatrix}
g(k,1) - \sum_{i=2}^{k-1}\sum_{j=1}^{I} h(k+1-i, 2-j) f(i,j) \\
g(k,2) - \sum_{i=2}^{k-1}\sum_{j=1}^{I} h(k+1-i, 3-j) f(i,j) \\
\vdots \\
g(k,N) - \sum_{i=2}^{k-1}\sum_{j=1}^{I} h(k+1-i, N+1-j) f(i,j) \\
p_k
\end{pmatrix}
$$

for $k = 2, 3, \cdots, M-1$. The determinant of the simultaneous linear equations is independent to k and it holds

$$
\begin{vmatrix}
h(1,1) & \ddots & 0 & f(1,1) & \ddots & 0 \\
h(1,2) & \ddots & \vdots & f(1,2) & \ddots & \vdots \\
\vdots & \ddots & 0 & \vdots & \ddots & 0 \\
h(1,L) & \ddots & h(1,1) & f(1,I) & \ddots & f(1,1) \\
0 & \ddots & h(1,2) & 0 & \ddots & f(1,2) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \ddots & h(1,L) & 0 & \ddots & f(1,I) \\
0 & \dots & 0 & 1 & \dots & 1
\end{vmatrix}
$$

$$
= (-1)^I \sum_{n=1}^{L} h(1,n) \cdot
\begin{vmatrix}
h(1,1) & \ddots & 0 & f(1,1) & \ddots & 0 \\
h(1,2) & \ddots & \vdots & f(1,2) & \ddots & \vdots \\
\vdots & \ddots & \vdots & \vdots & \ddots & 0 \\
h(1,L-1) & \ddots & 0 & \vdots & \ddots & f(1,1) \\
h(1,L) & \ddots & h(1,1) & f(1,I) & \ddots & f(1,2) \\
0 & \ddots & h(1,2) & 0 & \ddots & \vdots \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \ddots & h(1,L) & 0 & \ddots & f(1,I)
\end{vmatrix}.
$$

Hence, we have Theorem 1. ■

In the theorem, the equation $\sum_{n=1}^{L} h(1,n) = 0$ means $p_1 = 0$ in the Step (iv). In this case, we cannot determine the pair of $h(k,n)$ and $f(k,n)$ uniquely. On the other case, that is the determinant is equal to zero, two matrices $h(1,n)$ and $f(1,n)$ become linear dependent in the mean of time lag. In both cases, we can decide the solvability in the Step (v) if we check the value p_1 obtained in the Step (iii) and two matrices of $h(1,n)$ and $f(1,n)$ obtained in the Step (iv).

Example 3 In the Step (v), let $k = 2$ and put

$$
H = \begin{pmatrix} a & b \\ h(2,1) & h(2,2) \end{pmatrix}
$$
$$
F = \begin{pmatrix} c & d & e \\ f(2,1) & f(2,2) & f(2,3) \end{pmatrix}.
$$

Then, the system of linear equations becomes

$$
\begin{pmatrix}
a & 0 & 0 & c & 0 \\
b & a & 0 & d & c \\
0 & b & a & e & d \\
0 & 0 & b & 0 & e \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}
\begin{pmatrix} f(2,1) \\ f(2,2) \\ f(2,3) \\ h(2,1) \\ h(2,2) \end{pmatrix}
=
\begin{pmatrix} g(2,1) \\ g(2,2) \\ g(2,3) \\ g(2,4) \\ p_2 \end{pmatrix}.
$$

By the expansion of the determinant for the bottom row to obtain the small determinants, we get

$$
\begin{vmatrix}
a & 0 & 0 & c & 0 \\
b & a & 0 & d & c \\
0 & b & a & e & d \\
0 & 0 & b & 0 & e \\
0 & 0 & 0 & 1 & 1
\end{vmatrix}
$$
$$
= -\begin{vmatrix} a & 0 & 0 & 0 \\ b & a & 0 & c \\ 0 & b & a & d \\ 0 & 0 & b & e \end{vmatrix}
+ \begin{vmatrix} a & 0 & 0 & c \\ b & a & 0 & d \\ 0 & b & a & e \\ 0 & 0 & b & 0 \end{vmatrix}
$$
$$
= -a \begin{vmatrix} a & 0 & c \\ b & a & d \\ 0 & b & e \end{vmatrix}
- b \begin{vmatrix} a & 0 & c \\ b & a & d \\ 0 & b & e \end{vmatrix}
$$
$$
= -(a+b) \begin{vmatrix} a & 0 & c \\ b & a & d \\ 0 & b & e \end{vmatrix}.
$$

Therefore, if we put $a = b = c = e = 1, d = 2$, we cannot solve the system of linear equations because it becomes linear dependent.

5. Conclusions

In this paper, we discussed a property of our proposed factorization method for polynomials using the inverse z-transformation. The method solves the factorization for polynomials by reducing the problem of two dimensional convolution to the problem of one dimensional convolution and solving some systems of linear equations. In the method, we cannot always obtain factorization for some polynomials because simultaneous linear equations generated in the method are not always linear independent. So, we showed the condition when such a bad case happens. In the future work, the improvement of our method such that it can solve factorization even in the bad case is remained.

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