Conditions and estimation of the number of optimal solutions in nonlinear continuous optimization problems

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Abstract—We show more simpler representations of optimal solutions using level set and two conditions for an unconstrained optimization problem: 1) a necessary and sufficient optimality for a Morse function and 2) an existence condition of an optimal solution for a polynomial objective function. We also estimate the number of solutions for the following two kinds of objective functions: 1) univariate polynomial functions and 2) separable functions.

1. Introduction

A continuous optimization problem,"minimize an objective function $f(\boldsymbol{x}) \equiv f(x_1, x_2, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$, subject to $\boldsymbol{x} \in S$," has been applied to many fields. In cases in which objective functions are convex functions, many theoretical results have been obtained[1, 7].

Many studies have also been carried out for problems with nonconvex or multimodal objective functions. Most of those studies have been to proposing algorithms and to investigating behavior of algorithms. However, theoretical studies for these problems have been insufficient than for convex optimization problems. For example, it has not even been defined the number of modalities for multimodal optimization problems has not even been defined. Demidenko[3] investigated the basic properties of nonconvex or multimodal problems. However, it is not take no account of existence of flat regions in a problem. For optimization problems in which flat regions exists, we have proposed the local minimal values set (l.m.v.s.) as a new definition of optimal solutions, and we defined the number of modalities as the number of connected components[5].

In this paper, we show simpler definitions of local optima by level sets than previous definitions[5]. Next, show two conditions in unconstrained optimization problems: 1) a necessary and sufficient optimality condition for Morse functions, and 2) an existence condition of an optimal solution for a polynomial function. The number of solutions for the each of two kinds of functions: 1) univariate polynomial functions and 2) separable functions, is also estimated.

The remainder of the paper is organized as follows. An optimization problem and definitions of (connected) level sets are shown as preliminaries in Sect. 2. In Sect. 3, simpler definitions of sets of local optima using level sets and optimality conditions for an unconstrained optimization problem are presented. Existence conditions of optimal solutions for polynomial objective functions are presented in Sect. 4. Finally, concluding remarks are given in Sect. 5.

2. Preliminaries

2.1. Optimization problem and its assumptions

A continuous optimization problem with an objective function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ and a constraint $S \subset \mathbb{R}^n$ is formulated as follows:

$$\text{(P)} \begin{cases} \text{minimize (min.)} & \tilde{f}(\boldsymbol{x}) \equiv \tilde{f}(x_1, x_2, \ldots, x_n), \\ \text{subject to (s.t.)} & \boldsymbol{x} \in S \subset \mathbb{R}^n. \end{cases}$$

In this problem, we assume that S is a compact and connected set and that function $\tilde{f}$ is continuous.

For studying the problem (P) as a unconstrained problem, we define the following objective function f.

Definition 1 An extended real-valued function $f : \mathbb{R}^n \to (-\infty, +\infty]$ for the objective function $\tilde{f}$ in problem (P) is defined as

$$f(\boldsymbol{x}) = \begin{cases} \tilde{f}(\boldsymbol{x}), & \boldsymbol{x} \in S; \\ +\infty, & \boldsymbol{x} \notin S. \end{cases} \quad (1)$$

2.2. Definitions of level set and connected level set

First, we define three kinds of level sets that are determined by a level value of function.

Definition 2 *3 Level set*: *A level set* $L^{\leq}(\alpha) \subset \mathbb{R}^n$, a *strictly level set* $L^{<}(\alpha) \subset \mathbb{R}^n$ and an *equal level set* $L^{=}(\alpha) \subset \mathbb{R}^n$ at a level $\alpha = f(\boldsymbol{x}) \in \mathbb{R}$ are defined, respectively, as

$$L^{\leq}(\alpha) = \{ \boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) \leq \alpha \}, \quad (2)$$

$$L^{<}(\alpha) = \{ \boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) < \alpha \}, \quad (3)$$

$$L^{=}(\alpha) = \{ \boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) = \alpha \}. \quad (4)$$

In addition, five kinds of connected level sets including a point are defined.

Definition 3 Connected level set of $\boldsymbol{x}$:

- *Connected level set etc.*: $L_c^{\leq}(\alpha; \boldsymbol{x})$ and $L_c^{\leq}(f(\boldsymbol{x}))$.
  The *connected component* of $L^{\leq}(\alpha)$ that includes $\boldsymbol{x}$ is called a *connected level set* and is denoted by $L_c^{\leq}(\alpha; \boldsymbol{x})$. In case in which the level value is $\alpha = f(\boldsymbol{x})$, its set is denoted by

$$L_c^{\leq}(f(\boldsymbol{x})) \equiv L_c^{\leq}(f(\boldsymbol{x}); \boldsymbol{x}) = L_c^{\leq}(\alpha; \boldsymbol{x}), \ (\text{at } \alpha = f(\boldsymbol{x})). \quad (5)$$

Similarly, a *connected equal level set* with level $\alpha = f(\boldsymbol{x})$ at point $\boldsymbol{x}$ is denoted as follows.

- *connected equal level set* with level $f(\boldsymbol{x})$ of $\boldsymbol{x}$:

$$L_c^{=}(f(\boldsymbol{x})) \equiv L_c^{=}(f(\boldsymbol{x}); \boldsymbol{x}) = L_c^{=}(\alpha; \boldsymbol{x}), \ (\text{at } \alpha = f(\boldsymbol{x})). \quad (6)$$

Since *connected strictly level set* $L_c^{<}(\alpha; \boldsymbol{x})$ at $\alpha = f(\boldsymbol{x})$ cannot include point $\boldsymbol{x}$, it is defined as follows.

- *connected strictly level set* with level $f(\boldsymbol{x})$ of $\boldsymbol{x}$:

$$L_c^{<}(f(\boldsymbol{x})) \equiv L_c^{<}(f(\boldsymbol{x}); \boldsymbol{x}) = L_c^{\leq}(f(\boldsymbol{x})) \setminus L_c^{=}(f(\boldsymbol{x})). \quad (7)$$

3. Simpler definitions of optimal solutions and optimality conditions in an unconstrained optimization problem

3.1. Simpler definitions of sets of optimal solutions using (connected) level sets

We define a set of (global) minima and sets of four kinds of local minimal solutions using (connected) level sets.

Definition 4 A set of (global) minima and four kinds of local minimal solution (local minima, sets of local minimal values, strictly local minima and isolated local minima) are denoted by $\boldsymbol{X}_{**}, \boldsymbol{X}_{*}, \boldsymbol{X}_{*}^{\ell}, \boldsymbol{X}_{*}^{s}, \boldsymbol{X}_{*}^{i}$, respectively, and are formulated as follows.

$$\boldsymbol{X}_{**} = \{\, \boldsymbol{x}_{**} \mid L^{<}(f(\boldsymbol{x}_{**})) = \emptyset \,\} \quad (8)$$

$$\boldsymbol{X}_{*} = \{\, \boldsymbol{x}_{*} \mid \exists \boldsymbol{x}_{*} \in S,\ \exists \delta_1 > 0,\ \forall \boldsymbol{x} \in B(\boldsymbol{x}_{*}, \delta_1);\ f(\boldsymbol{x}_{*}) \leq f(\boldsymbol{x}) \,\} \quad (9)$$

$$\boldsymbol{X}_{*}^{\ell} = \{\, \boldsymbol{x}_{*}^{\ell} \mid L_c^{<}(f(\boldsymbol{x}_{*}^{\ell})) = \emptyset \,\} \quad (10)$$

$$\boldsymbol{X}_{*}^{s} = \{\, \boldsymbol{x}_{*}^{s} \mid L_c^{=}(f(\boldsymbol{x}_{*}^{s})) = \{\boldsymbol{x}_{*}^{s}\} \} \quad (11)$$

$$\boldsymbol{X}_{*}^{i} = \{\, \boldsymbol{x}_{*}^{i} \mid \boldsymbol{x}_{*}^{i} \in \boldsymbol{X}_{*}^{s} \text{ is isolated.} \,\} \quad (12)$$

From the above definitions, inclusion relations among these sets are easily derived as follows.

$$\boldsymbol{X}_{**} \subset \boldsymbol{X}_{*}^{\ell} \subset \boldsymbol{X}_{*}, \quad \boldsymbol{X}_{*}^{i} \subset \boldsymbol{X}_{*}^{s} \subset \boldsymbol{X}_{*}^{\ell} \subset \boldsymbol{X}_{*} \quad (13)$$

3.2. Previous necessary optimality conditions and sufficient optimality condition

We investigate the following *unconstrained optimization problem* that removes the constraint from problem (P) as follows:

$$(\text{Pu}) \quad \min. \quad \tilde{f}(x) \equiv f(\boldsymbol{x}), \quad (14)$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is (twice) continuously differentiable at any minimal solution. In this problem, three optimality conditions have been shown[2].

Theorem 5 (*First order necessary optimality condition*): If $\boldsymbol{x}_{*}$ is a local minimum, and f is continuously differentiable at point $\boldsymbol{x}_{*}$, the following equation holds[2].

$$\nabla f(\boldsymbol{x}_{*}) = \boldsymbol{0} \Longleftrightarrow \frac{\partial f}{\partial x_i}(\boldsymbol{x}_{*}) = 0,\ (i = 1, 2, \ldots, n). \quad (15)$$

The point $\boldsymbol{x}_{*}$ that satisfies $\nabla f(\boldsymbol{x}_{*}) = \boldsymbol{0}$ is called a *stationary point*.

Since any maximal point is also a stationary point, the following corollary holds.

Corollary 6 If $\boldsymbol{x}_{*}$ is the maximum of an unconstrained optimization problem and f is differentiable at point $\boldsymbol{x}_{*}$, then Eq. 15) holds.

Next, if f is twice differentiable around $\boldsymbol{x}_{*}$, then the following *second order optimality conditions* holds[2].

Theorem 7 (*Second order necessary optimality condition*):
If the point $\boldsymbol{x}_{*}$ is a local optimum of (Pu) and f is differentiable around $\boldsymbol{x}_{*}$, then the following equation holds[2],

$$\begin{cases} \nabla f(\boldsymbol{x}_{*}) = \boldsymbol{0} \\ \forall \boldsymbol{y} \neq 0; \quad \boldsymbol{y}^T \nabla^2 f(\boldsymbol{x}_{*}) \boldsymbol{y} \geq 0, \end{cases} \quad (16)$$

where $\nabla^2 f(\boldsymbol{x}_{*})$ is the Hesse matrix at point $\boldsymbol{x}_{*}$.

Moreover, if f is twice differentiable around $\boldsymbol{x}_{*}$, then the following *second order optimality conditions* holds.

Theorem 8 *(Second order sufficient optimality condition)*: Suppose that the point $\boldsymbol{x}_{*}$ is a local optimum of the problem (Pu) and f is twice differentiable around $\boldsymbol{x}_{*}$. Then the equation holds.

$$\begin{cases} \nabla f(\boldsymbol{x}_{*}) = \boldsymbol{0} \\ \forall \boldsymbol{y} \neq \boldsymbol{0}; \quad \boldsymbol{y}^T \nabla^2 f(\boldsymbol{x}_{*}) \boldsymbol{y} > 0. \end{cases} \quad (17)$$

3.3. Necessary and sufficient optimality condition

A stationary point $\boldsymbol{x}_{*}$ is called *non-degenerate* if and only if the Hesse matrix is non-singular as follows.

$$\det \left| \nabla^2 f(\boldsymbol{x}_{*}) \right| \neq 0 \iff \forall \boldsymbol{y} \neq \boldsymbol{0},\ \boldsymbol{y}^T \nabla^2 f(\boldsymbol{x}_{*}) \boldsymbol{y} \neq 0. \quad (18)$$

A Morse function is defined by focusing on such a non-degenerated stationary point.

Definition 9 Morse function: f is called a *Morse function* such that all stationary points of f are non-degenerated.[6].

Moreover, a Morse function has the following property.

Property 10 All stationary points of a Morse function are isolated[6].

Theorem 11 (*Second order necessary and sufficient (local) optimality condition for a Morse function*):
If the point $\boldsymbol{x}_{*}$ is a local minimum (optimum) of the problem (Pu) with Morse function f, and suppose that f is twice continuously differentiable around $\boldsymbol{x}_{*}$. Then, the following necessary and sufficient optimality condition holds.

$$\begin{cases} \nabla f(\boldsymbol{x}_{*}) = \boldsymbol{0} \\ \forall \boldsymbol{y} \neq \boldsymbol{0}; \quad \boldsymbol{y}^T \nabla^2 f(\boldsymbol{x}_{*}) \boldsymbol{y} > 0. \end{cases} \quad (19)$$

Proof Since all stationary points of a Morse function are non-degenerate, if f is a Morse function, then Eq. (18) holds. Thus, the equality not holds at the second inequality:$\boldsymbol{y}^T \nabla^2 f(\boldsymbol{x}_{*}) \boldsymbol{y} \geq 0$ of Eq. (16). In that case, Eq. (16) is become to same to Eq. (17) of **Theorem 8**. ■

4. Existence conditions of a minimum

In the previous section, we show that if f is a Morse objective function of the unconstrained problem then an isolated minimum exist. However, for example $f(x)=x^4$ is not a Morse function because $\nabla^2 f(\boldsymbol{x}_*)=f(0)=0$ at the unique minimum point: $\boldsymbol{x}_* = 0$. but the function have unique minimum at $x=0$. In this section, the existence conditions of such an optimal solution are given.

4.1. Existence conditions of a minimum for a multivariate function

For a real-valued continuous function, the following Weierstrass's *extreme value theorem* is very important and is available.

Theorem 12 *(*Weierstrass's theorem*):*
If function $f: S \to \mathbb{R}$ is continuous on compact set S, then at least one minimum exists.

From the theorem, It can be considered that level set $L^{\leq}(\alpha)=\{\boldsymbol{x}\in\mathbb{R}^n \mid f(\boldsymbol{x})\leq\alpha\}$ is always compact at any level $\alpha\in\mathbb{R}$. One kind of functions that satisfied such a condition is shown as follows.

Definition 13 A continuous objective function $f:\mathbb{R}^n\to\mathbb{R}$ is called *coercive* if

$$\lim_{\|\boldsymbol{x}\|\to\infty} f(\boldsymbol{x}) = +\infty. \tag{20}$$

In a case in which f is not coercive (e.g., $f(x)=1/(x^2+1)$), Demidenko[3] presented the following definition.

Definition 14 An *upper existence level* $\overline{L}_f$ of continuous bounded-from-below function f is defined as

$$\overline{L}_f = \lim_{r\to\infty}\inf_{\|\boldsymbol{x}\|\geq r} f(\boldsymbol{x}). \tag{21}$$

Demidenko showed the following theorem[3] in a case in which $\overline{L}_f$ is finite.

Theorem 15 *If $\boldsymbol{x}_0\in\mathbb{R}^n$ is a point and $f(\boldsymbol{x}_0)<\overline{L}_f$, then level $L^{\leq}(f(\boldsymbol{x}_0))$ is compact and a minimum exists on $\mathbb{R}^n$.*

4.2. Existence condition of a minimum for univariate polynomial functions

To determine whether a p-th degree univariate polynomial function $f:\mathbb{R}\to\mathbb{R}$ with real coefficients,

$$f(x) = a_p x^p + a_{p-1}x^{p-1} + \cdots + a_1 x + a_0,\ (a_p\neq 0) \tag{22}$$

is coercive or not, we estimate the following function values by $|x|\to\infty$:

$$\lim_{|x|\to\infty} f(x) = \begin{cases} +\infty, & p=2m,\ a_p>0 \text{ (coercive)},\\ -\infty, & p=2m,\ a_p<0,\\ \pm\infty, & p=2m-1,\ (m=1,2,\ldots),\end{cases} \tag{23}$$

and we have the following property.

Property 16 The necessary and sufficient existence condition of a minimum for a p-th degree polynomial function $f:\mathbb{R}\to\mathbb{R}$ is that the degree is even and its coefficient is positive, that is $p=2m\ (m=1,2,\ldots),\ a_p>0$.

4.3. Existence condition of a minimum for multivariate polynomial functions

By using $(k_1+1)(k_2+1)\cdots(k_n+1)$-coefficients $a_{\rho_1\rho_2\ldots\rho_n}$ $(0\leq \rho_i\leq k_i$ (positive integer), $i=1,2,\ldots,n)$ including at least one nonzero, polynomials of n-variables are represented as

$$f(x_1,x_2,\ldots,x_n)=\sum_{\substack{\rho_i\in[0,k_i]\\ i\in[1,n]}} a_{\rho_1\rho_2\ldots\rho_n} x_1^{\rho_1}x_2^{\rho_2}\cdots x_n^{\rho_n}. \tag{24}$$

In addition, let *monomial* and *its coefficient* be

$$\begin{cases} \boldsymbol{x}^{\boldsymbol{\rho}} \equiv x_1^{\rho_1}x_2^{\rho_2}\cdots x_n^{\rho_n} & \cdots \text{ (monomial)},\\ a_{\boldsymbol{\rho}} \equiv a_{\rho_1\rho_2\ldots\rho_n} & \cdots \text{ (coefficient)}.\end{cases} \tag{25}$$

By using the above notations, original Eq. (24) of polynomials is simplified as follows.

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{\rho}\in\mathcal{S}_f} a_{\boldsymbol{\rho}}\boldsymbol{x}^{\boldsymbol{\rho}}, \tag{26}$$

where $\mathcal{S}_f=\{\boldsymbol{\rho} : \text{non-negative integer} \mid a_{\boldsymbol{\rho}}\neq\boldsymbol{0}\}$, and the degree $\deg(a_{\boldsymbol{\rho}}\boldsymbol{x}^{\boldsymbol{\rho}})$ of each term $a_{\boldsymbol{\rho}}\boldsymbol{x}^{\boldsymbol{\rho}}$ and degree $\deg(f)$ of f are given as follows:

$$\begin{cases} \deg(a_{\boldsymbol{\rho}}\boldsymbol{x}^{\boldsymbol{\rho}}) = \left\{\sum_{i=1}^n \rho_i \;\middle|\; \boldsymbol{\rho}\in\mathcal{S}_f\right\}\\ \deg(f) = \max\{\deg(a_{\boldsymbol{\rho}}\boldsymbol{x}^{\boldsymbol{\rho}}) \;\big|\; \boldsymbol{\rho}\in\mathcal{S}_f\},\\ \quad \text{where } \boldsymbol{\rho}\equiv\{\rho_1,\rho_2,\ldots,\rho_n\}.\end{cases} \tag{27}$$

The definitions of two terms by dividing into two categories term $\boldsymbol{\rho}$ and of related set to two terms.

Definition 17 It is called a *pure term* if only one element of $\boldsymbol{\rho}$ is nonzero and others are 0, and otherwise it is called a *mixed term*. These are represented as follows.

$$\begin{cases} \rho_i\neq 0\,(\exists i\in I^n),\ \rho_j=0\,(j\in I^n\setminus\{i\}) & \text{pure term},\\ \text{otherwise} & \text{mixed term},\\ \quad \text{where } I^n=\{1,2,\ldots,n\} \end{cases} \tag{28}$$

where the set of coefficients and degrees of pure term are denoted by T_p, C_p and P_p respectively, and coefficients and degrees of pure term are denoted by T_m, C_m and P_m, respectively. Since the pure term consists of only one variable, the pure term in $\boldsymbol{a}_\rho\boldsymbol{x}^{\boldsymbol{\rho}}$ is represented as follows.

$$\forall \boldsymbol{a}_\rho\boldsymbol{x}^{\boldsymbol{\rho}}\in T_p;\ \ a_{\rho_i}x_i^{\rho_i}\equiv\boldsymbol{a}_\rho\boldsymbol{x}^{\boldsymbol{\rho}},\ i\in I^n,\ \rho_i\in\{1,2,\ldots,k_i\}. \tag{29}$$

From the above, it is possible to derive the following property for the existence condition of a minimum.

Theorem 18 Let the min. degree:$\underline{\deg}(T_p)$ in pure terms and the max. degree:$\overline{\deg}(T_m)$ in mixed terms be

$$\begin{cases} \underline{\deg}(T_p) = \min_{i\in I^n}\{\overline{\rho}_i\},\\ \overline{\deg}(T_m) = \max\{\deg(\boldsymbol{a}_\rho\boldsymbol{x}^{\boldsymbol{\rho}}) \mid \boldsymbol{a}_\rho\boldsymbol{x}^{\boldsymbol{\rho}}\in T_m\},\end{cases} \tag{30}$$

where $\overline{\rho}_i$ is given as the maximum degree of terms consisting only x_i as follows.

$$\overline{\rho}_i \equiv \max_{\rho_i\in[1,k_i]}\{\deg(a_{\rho_i}x_i^{\rho_i})\}. \tag{31}$$

If a multivariate polynomial $f:\mathbb{R}\to\mathbb{R}^n$ satisfies the following equation:

$$\begin{cases} \underline{\deg}(T_p) > \overline{\deg}(T_m),\\ \overline{\rho}_i = 2m\,(m: \text{natural number},\ \overline{\rho}_i\in P_p),\\ a_{\overline{\rho}_i}>0\ (a_{\overline{\rho}_i}\in C_p), \qquad (i=1,2,\ldots,n),\end{cases} \tag{32}$$

then f is coercive and has a minimum.

Proof (—Omission—)

Example 19 We think the following minimization problem[4] with objective function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$\text{min. } f(\boldsymbol{x}) \equiv \frac{1}{3}x_1^6 - 2.1x_1^4 + 4x_1^2 + x_1x_2 + 4x_2^4 - 4x_2^2. \quad (33)$$

In the objective function f, pure terms are $(1/3)x_1^6$, $-2.1x_1^4$, $4x_1^2$, $4x_2^4$, $-4x_2^2$. From Eq. (30), the max. degree of x_1 in pure terms is $\overline{\rho}_1 = 6$, the max. degree of x_2 in pure terms is $\overline{\rho}_2 = 4$, and the min. degree in pure terms is $\underline{\deg}(T_p) = \min\{6, 4\} = 4$. On the other hand, From the mixed term is only x_1x_2, the max. degree of mixed term is $\overline{\deg}(T_m) = \min\{2\} = 2$. Thus, f satisfies the first condition in Eq. (32). Moreover, since all of $\overline{\rho}_i$ $(i = 1, 2)$ are even numbers, coefficients of the max. degree in x_1 and x_2 are $1/3 > 0$ and $4 > 0$, and the second and third conditions of Eq. (32) are also satisfied. Therefore, the f has a minimum.

5. Estimation of the number of isolated local minima

5.1. Relation between local minima and local maxima for univariate functions

Property 20 $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ and let its numbers of local minima x_*^i $(i = 1, 2, \dots)$ and of local maxima $\overline{x}_*$ $(i = 1, 2, \dots)$ in ascending order be M and $\overline{M}$, then the following four equations holds.

$$\begin{cases} a \le \overline{x}_*^1 < x_*^1 < \overline{x}_*^2 < x_*^2 < \cdots < \overline{x}_*^M < x_*^M < \overline{x}_*^{M+1} \le b & 1), \\ a \le \overline{x}_*^1 < x_*^1 < \overline{x}_*^2 < x_*^2 < \cdots < \overline{x}_*^M < x_*^M \le b & 2), \\ a \le x_*^1 < \overline{x}_*^1 < x_*^2 < \overline{x}_*^2 < \cdots < \overline{x}_*^{M-1} < x_*^M < \overline{x}_*^M \le b & 3), \\ a \le x_*^1 < \overline{x}_*^1 < x_*^2 < \overline{x}_*^2 < \cdots < \overline{x}_*^{M-1} < x_*^M \le b & 4). \end{cases} \quad (34)$$

In addition, the following equation between the number of local minima M and the number of local maxima $\overline{M}$ holds.

$$\begin{cases} \overline{M} - 1 \le M \le \overline{M} + 1 \\ 2M - 1 \le M + \overline{M} \le 2M + 1. \end{cases} \quad (35)$$

5.2. Estimation of the number of local minima for univariate functions and separable functions

The following property can be given in univariate polynomial functions of degree p.

Property 21 If a univariate polynomial function $f : \mathbb{R} \to \mathbb{R}$ of degree p has a minimum in an unconstrained optimization problem (Pu), the number of local minima M of f is $M \le p/2$ (p : odd number) and all local minima are isolated.

Proof (—omission—)

A separable function $f : \mathbb{R}^n \to \mathbb{R}$ is formulated as

$$f(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} f_i(x_i). \quad (36)$$

The next property with respect to number of local minima in an unconstrained problem (Pu) with a separable function will be established.

Property 22 Let the number of local minima of each element function $f_i : \mathbb{R} \to \mathbb{R}$ in separable function $f : \mathbb{R}^n \to \mathbb{R}$ given by eq. (36) be M_i. Then the number of local minima M of f is estimated as follows.

$$M = M_1 \times M_2 \times \cdots \times M_n = \prod_{i=1}^{n} M_i. \quad (37)$$

Proof (—omission—)

If the region $\mathbb{R}^n$ changes into inner of hyper box region $D^n \equiv \prod_{i=1}^{n} [a_i, b_i]$, then the above property also holds,

Example 23 Let a separable function f that is sum of f_1 and f_2 be as follows:

$$f(x_1, x_2) = \{f_1(x_1)\} + \{f_2(x_2)\} = \left\{\frac{1}{3}x_1^6 - 2.1x_1^4 + 4x_1^2 + x_1\right\} + \left\{4x_2^4 - 4x_2^2 + x_2\right\}.$$

Since the upper bounds(M_1, M_2) of the number of local minima on f_1 and f_2 are $M_1 = 3$ and $M_2 = 2$, the upper bound of the number of local M is $M = M_1 \times M_2 = 3 \times 2 = 6$.

6. Conclusions

In this paper, we showed two conditions: 1) a necessary and sufficient optimality condition for an unconstrained optimization problem with a Morse function and 2) an existence condition of an optimal solution for a polynomial objective function. We also estimated the number of solutions for the following two kinds of functions: 1) univariate polynomial functions and 2) separable functions.

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