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Continuous-Time Method to Plan Volumetric Modulated Arc Therapy

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Abstract—Volumetric modulated arc therapy (VMAT) is a treatment method that irradiates X-rays to target tissues within human body while continuously rotating a gantry. Since the intensity of X-rays must be controlled throughout the rotation, large changes of X-ray intensity are not desirable. The problem of finding a VMAT plan can be formulated as an inverse problem and solved with a dynamical system. In this paper, to find VMAT plans, we proposed an objective function with regularization related to the intensity of X-rays and a dynamical system to minimize the objective function. Our experimental results showed that the proposed system worked well for a toy problem on VMAT.

1. Introduction

Intensity modulated radiation therapy (IMRT) [1] is a treatment method that irradiates X-rays to cancer cells within human body. In IMRT, the gantry that radiates X-rays repeats the operation of pausing at predetermined angles and irradiating X-rays that are controlled with multi-leaf collimator (MLC) to cancer sells. On the other hand, volumetric modulated arc therapy (VMAT) [2] that is an IMRT can reduce the treatment irradiation time because the gantry irradiates X-rays while rotating. However, the shape of the MLC needs to be quickly controlled while rotating.

The problem of planning IMRT can be formulated as an inverse problem and solved with a dynamical system [3]. Although it is desired for the change of X-ray intensity during the rotation in VMAT to be small from the viewpoint of MLC control, its change is not considered in the formulation of the inverse problem.

In this paper, to find VMAT plans, we propose an objective function with regularization related to the intensity of X-rays, which corresponds to the total variation [4] of X-ray intensity. We describe a dynamical system to minimize the proposed objective function and experimental results for a toy problem on VMAT.

2. Problem Description and Conventional System

Now,we are going to explain the problem of IMRT. In computer tomographic images, we define the regions of planning target volumes (PTVs) as cancer sells and organs at risk (OARs) as the healthy tissues. We set the lower and upper limits of doses for each PTV and the upper limit of doses for each OAR. While the doses for the other regions are not defined. Intensity modulated X-rays are irradiated so that prescribed doses at PTVs and OARs are satisfied.

According to the set-up, let us mathematically formulate the problem of planning IMRT. Let $\boldsymbol{w} \in (0, \gamma)^J$ as a vector of unknown variables to satisfy X-ray intensity

$$\boldsymbol{D} = \boldsymbol{K}\boldsymbol{w} \tag{1}$$

where γ is the upper limit of X-ray intensity, $\boldsymbol{D} \in \mathbb{R}_+^I$ corresponds to prescribed dose at each voxel, $\boldsymbol{K} \in \mathbb{R}_+^{I\times J}$ is the dose transfer matrix, I and J are the numbers of voxels and X-ray paths, and $\mathbb{R}_+$ represents the non-negative real numbers. If the number of voxels at PTVs and OARs is I_1 and I_2, i.e., $(I_1 + I_2 = I)$, respectively, then $\boldsymbol{D}$ consists of $\boldsymbol{D}_{\rm PTV} \in \mathbb{R}_+^{I_1}$ and $\boldsymbol{D}_{\rm OAR} \in \mathbb{R}_+^{I_2}$, and $\boldsymbol{K}$ consists of $\boldsymbol{K}_{\rm PTV} \in \mathbb{R}_+^{I_1\times J}$ and $\boldsymbol{K}_{\rm OAR} \in \mathbb{R}_+^{I_2\times J}$, we define

$$\boldsymbol{D} = \begin{pmatrix} \boldsymbol{D}_{\rm PTV} \\ \boldsymbol{D}_{\rm OAR} \end{pmatrix} \text{ and } \boldsymbol{K} = \begin{pmatrix} \boldsymbol{K}_{\rm PTV} \\ \boldsymbol{K}_{\rm OAR} \end{pmatrix}. \tag{2}$$

Let $D^0_{\rm PTV}$ and $D^1_{\rm PTV}$ correspond to the lower and upper limits of doses for PTV, and $D^0_{\rm OAR}$ expresses the upper limit of doses for OAR, $(0 < D^0_{\rm PTV} < D^1_{\rm PTV}$ and $D^0_{\rm OAR} > 0)$. We also define the projection P as

$$P : \mathbb{R}_+^I \to \mathbb{R}_+^I; \boldsymbol{D} = \begin{pmatrix} \boldsymbol{D}_{\rm PTV} \\ \boldsymbol{D}_{\rm OAR} \end{pmatrix} \mapsto P(\boldsymbol{D}), \tag{3}$$

where

$$(P(D))_{i_1} = \begin{cases} D^1_{\rm PTV}, & \text{if } (D_{\rm PTV})_{i_1} > D^1_{\rm PTV} \\ (D_{\rm PTV})_{i_1}, & \text{if } D^1_{\rm PTV} \geq (D_{\rm PTV})_{i_1} \geq D^0_{\rm PTV} \\ D^0_{\rm PTV}, & \text{otherwise} \end{cases} \tag{4}$$

$$(P(D))_{I_1+i_2} = \begin{cases} (D_{\rm OAR})_{i_2}, & \text{if } (D_{\rm OAR})_{i_2} \leq D^0_{\rm OAR} \\ D^0_{\rm OAR}, & \text{otherwise} \end{cases} \tag{5}$$

for $i_1 = 1, 2, \ldots, I_1$ and $i_2 = 1, 2, \ldots, I_2$.

The IMRT planning problem can be formulated as a problem to minimize an objective function defined as

$$F(\boldsymbol{w}) = \frac{1}{2}\left\|\boldsymbol{K}\boldsymbol{w} - P(\boldsymbol{K}\boldsymbol{w})\right\|_2^2 \tag{6}$$

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where $\|\cdot\|_p$ represents the p -norm. This can be replaced into an initial-value problem of the ordinary differential equation

$$\frac{d\boldsymbol{w}}{dt} = -\boldsymbol{W}\left(\boldsymbol{U} - \gamma^{-1}\boldsymbol{W}\right)\boldsymbol{K}^{\top}(\boldsymbol{K}\boldsymbol{w} - P(\boldsymbol{K}\boldsymbol{w})) \tag{7}$$

where $\boldsymbol{W} = \mathrm{diag}(\boldsymbol{w})$, the superscript $\top$ means transpose of a matrix, $\boldsymbol{U}$ represents the identity matrix, and initial values are set to $(0, \gamma)$. Hence, by calculating the trajectory of a solution to Eq. (7), we can find an IMRT plan that corresponds to a local minimizer of Eq. (6) [3]. However, in general, the method cannot find VMAT plans such that the change of X-ray intensity during rotation is small.

3. Proposed System

To be able to find suitable plans for VMAT, we introduce a regularization term into Eq.(6) as

$$H(\boldsymbol{w}) = \frac{1}{2}\left\|\boldsymbol{K}\boldsymbol{w} - P(\boldsymbol{K}\boldsymbol{w})\right\|_2^2 + \eta T(\boldsymbol{w}) \tag{8}$$

where $\eta > 0$ is a parameter that determines the effect of regularization; $T(\boldsymbol{w})$ corresponds to the total variation of the intensity of X-ray bundles at each angle and is defined as below. We now assume that m and n correspond to the number of X-ray paths at each angle. For all the X-ray paths at the kth angle, we define the bundle of X-ray paths as $\boldsymbol{W}_{g_k} = (\boldsymbol{w}_{(k-1)m+1}, \boldsymbol{w}_{(k-1)m+2}, \ldots, \boldsymbol{w}_{km})^{\top}$, and then, we obtain the distribution of the intensity of X-ray bundles as $\boldsymbol{W}_G = \left(\|\boldsymbol{W}_{g_1}\|_1, \|\boldsymbol{W}_{g_2}\|_1, \ldots, \|\boldsymbol{W}_{g_n}\|_1\right)^{\top}$. Therefore, the regularization term in Eq. (8) is defined as

$$T(\boldsymbol{w}) = \sum_{k=1}^{n-1} \left\| \|\boldsymbol{W}_{g_{k+1}}\|_1 - \|\boldsymbol{W}_{g_k}\|_1 \right\|_1. \tag{9}$$

To find a local minimizer of $H(\boldsymbol{w})$ that corresponds to a suitable treatment plan for VMAT, we propose a continuous-time dynamical system as

$$\frac{d\boldsymbol{w}}{dt} = -\boldsymbol{W}\left(\boldsymbol{U} - \gamma^{-1}\boldsymbol{W}\right)\left\{\boldsymbol{K}^{\top}(\boldsymbol{K}\boldsymbol{w} - P(\boldsymbol{K}\boldsymbol{w})) + \eta\frac{\partial T(\boldsymbol{w})}{\partial \boldsymbol{w}}\right\}. \tag{10}$$

Here ,the gradient of total variation, $\partial T(\boldsymbol{w})/\partial \boldsymbol{w}$, is efficiently calculated according to Kawamura's method [4].

4. Experimental Results and Discussion

To check the performance of the proposed system, we treated a toy problem using VMAT. It consists of a 3×3 pixel phantom image with two PTV and two OAR regions shown in Fig. 1 and the prescribed doses for each region is shown in Table1.

The conventional method using Eq. (7) with $w_j(0) = 0.01, \forall j$, was applied to the toy problem. We calculated the values of $\boldsymbol{w}(100)$ and obtained the dose volume histogram (DVH) and dose distribution map (DDM) as shown

Table 1: Prescribed doses

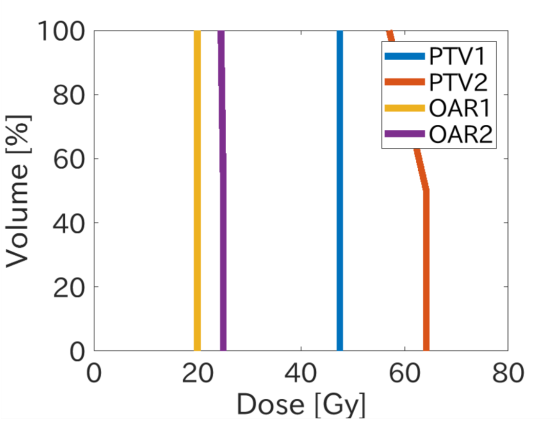
	D^0_{PTV}	D^1_{PTV}	D^0_{OAR}
PTV_1	47.5	53.5	—
PTV_2	57.0	64.2	—
OAR_1	—	—	20
OAR_2	—	—	25

PTV_1	OAR_1	PTV_2
	OAR_1	PTV_2
	OAR_2	OAR_2

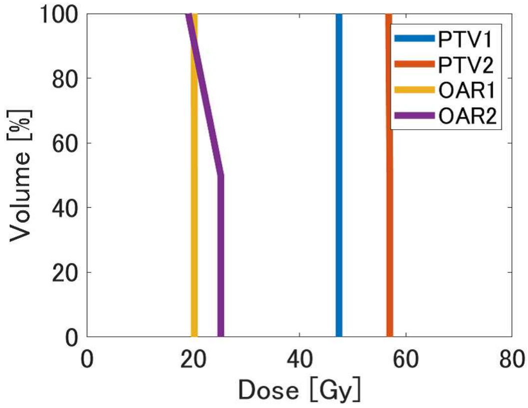
Figure 1: PTVs and OARs in 3×3 pixel phantom image

in Fig. 2(a) and 3(a). Based on the DVH graph, the abscissa expresses total irradiated dose and the ordinate represents the percentage volume corresponding to the irradiated dose for each region. Sence, each line shows the histogram of each region: PTV_1, PTV_2, OAR_1, and OAR_2. Based on the DVH, we can see that the 100% of PTV_1, OAR_1, and OAR_2 regions were irradiated with 47.5 Gy, 20.0 Gy, and around 25.0 Gy, of dose respectively; While 50% of PTV_2 region, 57.0 Gy of dose was applied and the irradiated dose of the remaining volume was 64.0 Gy. Therefore, all the irradiated doses satisfied the prescribed doses in Table 1. Based on the DDM, the irradiated dose at each voxel was displayed in correspondence with colors: gray, orange, and yellow colors correspond to low, middle, and high dose respectively. Note that the regions that are neither PTV nor OAR were shown in black color because the lower and upper limits of irradiated dose were undefined at the regions. Based on the DDM, the two PTVs were applied with high dose and low dose was applied to the two OARs. The DVH and DDM showed that the obtained treatment plan is suitable for IMRT. However, it is not appropriate as VMAT plans according to the results shown in Fig. 4(a). In the figure, the abscissa expresses irradiation angles and the ordinate represents the intensity of X-ray bundles, i.e., the height of each bar corresponds to the value of $\|\boldsymbol{W}_{g_k}\|_1, (k = 1, 2, \ldots, 360)$. Hence, the intensity of X-ray bundles drastically changed versus irradiation angles, which means that multi-leaf collimators need to move significantly, and besides, we need to control the movement quickly and accurately.

We set the initial value $w_j(0) = 0.01$ and applied the proposed method using Eq. (10) with $\eta = 0.1$ to the toy problem. We obtained a DVH, DDM, and the intensity of X-ray bundles versus irradiation angles are shown in Figs. 2(b), 3(b), and 4. The shape and position of each line in the DVH and the colors in the DDM were almost the same as the results obtained with the conventional method. The irradi-



(a) Conventional method



(b) Proposed method

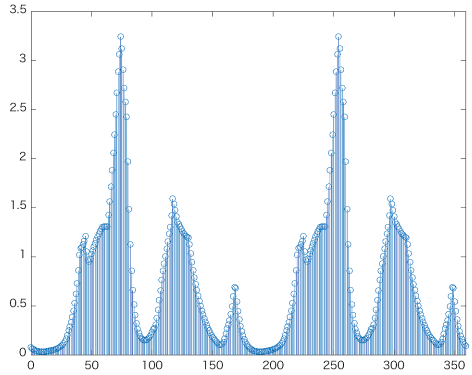
Figure 2: Dose volume histogram

(a) Conventional method

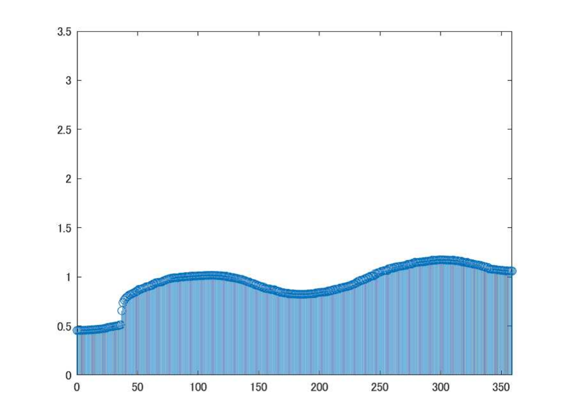
(b) Proposed method

Figure 3: Dose distribution map

ated doses at PTV_1, PTV_2, and OAR_1 were 47.5 Gy, 57.0 Gy, and 20.0 Gy respectively; 50% of OAR_2 region was irradiated with 20.4 Gy of dose and the remaining region was irradiated with 25.0 Gy of dose. The results showed that all the irradiated doses satisfied the prescribed doses in Table 1. In addition, the change of X-ray intensity versus irradiation angles in Fig. 4 was quite smaller compared to the results of the conventional method. Based on the results, the proposed method was able to produce a suitable treatment plan for VMAT.



(a) Conventional method



(b) Proposed method

Figure 4: Intensity of X-ray bundles versus irradiation angle

5. Conclusion

In this paper, to produse suitable treatment plans for VMAT, we proposed the objective function with the regularization related to the intensity of X-ray bundle and the dynamical system to solve it. Our experimental results for the toy problem showed that the proposed system worked well and was able to produce a suitable treatment plan for VMAT. Our future works are to analyze the local stability of equilibrium points observed in Eq. (10) and to apply the proposed method to VMAT problems consist of 512×512 CT images.

A. Gradient of total variation

As a simple example, we now consider a problem with six X-ray beams

$$\boldsymbol{w} = (w_1, w_2, \ldots, w_6) \tag{11}$$

that are divided into three groups (directions) as

$$\boldsymbol{W}_{g_1} = (w_1, w_2) \tag{12}$$

$$\boldsymbol{W}_{g_2} = (w_3, w_4) \tag{13}$$

$$\boldsymbol{W}_{g_3} = (w_5, w_6) \tag{14}$$

and then we obtain the distribution of the intensity of X-ray bundles

$$\boldsymbol{W}_G = \left(\|\boldsymbol{W}_{g_1}\|_1, \|\boldsymbol{W}_{g_2}\|_1, \|\boldsymbol{W}_{g_3}\|_1\right). \tag{15}$$

We note that these vectors are defined as row vectors to simplify the following derivation.

To calculate the total variation $T(\boldsymbol{w})$ in Eq.(9), we define an operator

$$\boldsymbol{C} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{16}$$

and then we have

$$\begin{aligned} T(\boldsymbol{w}) &= S(\boldsymbol{W}_G) \\ &= \|\boldsymbol{C} \cdot \boldsymbol{W}_G\|_1 \\ &= \left\| \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \|\boldsymbol{W}_{g_1}\|_1 \\ \|\boldsymbol{W}_{g_2}\|_1 \\ \|\boldsymbol{W}_{g_3}\|_1 \end{pmatrix} \right\|_1 \\ &= \left\| \begin{pmatrix} \|\boldsymbol{W}_{g_2}\|_1 - \|\boldsymbol{W}_{g_1}\|_1 \\ \|\boldsymbol{W}_{g_3}\|_1 - \|\boldsymbol{W}_{g_2}\|_1 \\ 0 \end{pmatrix} \right\|_1 . \end{aligned} \tag{17}$$

Therefore, the gradient of $T(\boldsymbol{w})$ can be calculated as

$$\begin{aligned} \frac{\partial T(\boldsymbol{w})}{\partial \boldsymbol{w}} &= \frac{\partial S(\boldsymbol{W}_G)}{\partial \boldsymbol{w}} = \frac{\partial S(\boldsymbol{W}_G)}{\partial \boldsymbol{W}_G} \cdot \frac{\partial \boldsymbol{W}_G}{\partial \boldsymbol{w}} \\ &= \begin{pmatrix} \frac{\partial S(\boldsymbol{W}_G)}{\partial \|\boldsymbol{W}_{g_1}\|_1} & \frac{\partial S(\boldsymbol{W}_G)}{\partial \|\boldsymbol{W}_{g_2}\|_1} & \frac{\partial S(\boldsymbol{W}_G)}{\partial \|\boldsymbol{W}_{g_3}\|_1} \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} \frac{\partial \|\boldsymbol{W}_{g_1}\|_1}{\partial w_1} & \frac{\partial \|\boldsymbol{W}_{g_1}\|_1}{\partial w_2} & \cdots & \frac{\partial \|\boldsymbol{W}_{g_1}\|_1}{\partial w_6} \\ \frac{\partial \|\boldsymbol{W}_{g_2}\|_1}{\partial w_1} & \frac{\partial \|\boldsymbol{W}_{g_2}\|_1}{\partial w_2} & \cdots & \frac{\partial \|\boldsymbol{W}_{g_2}\|_1}{\partial w_6} \\ \frac{\partial \|\boldsymbol{W}_{g_3}\|_1}{\partial w_1} & \frac{\partial \|\boldsymbol{W}_{g_3}\|_1}{\partial w_2} & \cdots & \frac{\partial \|\boldsymbol{W}_{g_3}\|_1}{\partial w_6} \end{pmatrix}. \end{aligned} \tag{18}$$

Here, the entries of the vector and matrix can be calculated with subgradients

$$\frac{d|z|}{dz} = \mathrm{sign}(z) = \begin{cases} 1 & z > 0 \\ -1 & z < 0 \\ 0 & z = 0 \end{cases} \tag{19}$$

as

$$\begin{aligned} &\frac{\partial S(\boldsymbol{W}_G)}{\partial \boldsymbol{W}_G} \\ &= \begin{pmatrix} -\mathrm{sign}\left(\|\boldsymbol{W}_{g_2}\|_1 - \|\boldsymbol{W}_{g_1}\|_1\right) \\ \mathrm{sign}\left(\|\boldsymbol{W}_{g_2}\|_1 - \|\boldsymbol{W}_{g_1}\|_1\right) - \mathrm{sign}\left(\|\boldsymbol{W}_{g_3}\|_1 - \|\boldsymbol{W}_{g_2}\|_1\right) \\ \mathrm{sign}\left(\|\boldsymbol{W}_{g_3}\|_1 - \|\boldsymbol{W}_{g_2}\|_1\right) \end{pmatrix}^\top \\ &= \left(\mathrm{Sign}\,(\boldsymbol{C} \cdot \boldsymbol{W}_G)^\top \boldsymbol{C}\right)^\top . \end{aligned} \tag{20}$$

In $\mathrm{Sign}(\boldsymbol{y})$, a sign function is applied to each element of vector $\boldsymbol{y}$. Moreover, we obtain

$$\frac{\partial \boldsymbol{W}_G}{\partial \boldsymbol{w}} = \begin{pmatrix} \mathrm{sign}(w_1) & 0 & 0 \\ \mathrm{sign}(w_2) & 0 & 0 \\ 0 & \mathrm{sign}(w_3) & 0 \\ 0 & \mathrm{sign}(w_4) & 0 \\ 0 & 0 & \mathrm{sign}(w_5) \\ 0 & 0 & \mathrm{sign}(w_6) \end{pmatrix}^\top \tag{21}$$

according to

$$\frac{\partial \|\boldsymbol{W}_{g_i}\|_1}{\partial w_j} = \begin{cases} 1 & \text{if } w_j > 0 \text{ and } w_j \in \boldsymbol{W}_{g_i} \\ -1 & \text{if } w_j < 0 \text{ and } w_j \in \boldsymbol{W}_{g_i} \\ 0 & \text{otherwise} \end{cases} . \tag{22}$$

Equation (21) is also represented as

$$\frac{\partial \boldsymbol{W}_G}{\partial \boldsymbol{w}} = (\boldsymbol{I}_3 \otimes \boldsymbol{u}_2) \circ \left(\boldsymbol{w} \otimes \boldsymbol{u}_3^\top\right) \tag{23}$$

where $\boldsymbol{I}_3$ is the identity matrix with 3×3 in which the row size corresponds to the number of directions; $\boldsymbol{u}_n$ expresses the row vector in which all the elements are one and n means the number of elements; the symbols, $\otimes$ and $\circ$, correspond to the Kronecker product and the Hadamard product. Therefore, from Eqs.(20) and (23), we finally obtain the algebraic formulation of $\partial T(\boldsymbol{w})/\partial \boldsymbol{w}$ as

$$\begin{aligned} \frac{\partial T(\boldsymbol{w})}{\partial \boldsymbol{w}} &= \frac{\partial S(\boldsymbol{W}_G)}{\partial \boldsymbol{W}_G} \cdot \frac{\partial \boldsymbol{W}_G}{\partial \boldsymbol{w}} \\ &= \left(\mathrm{Sign}\,(\boldsymbol{C} \cdot \boldsymbol{W}_G)^\top \cdot \boldsymbol{C}\right)^\top \cdot \left\{(\boldsymbol{I}_3 \otimes \boldsymbol{u}_2) \circ \left(\boldsymbol{w} \otimes \boldsymbol{u}_3^\top\right)\right\}. \end{aligned} \tag{24}$$

References

[1] S. X. Chang, et al., “Intensity modulation delivery techniques: “Step & shoot” MLCauto-sequence versus the use of a modulator,” *Med. Phys.*, vol. 27, no. 5, pp. 948–959, 2000.

[2] M. Teoh, et al., “Volumetric modulated arc therapy: a review of current literature and clinical use in practice,” *Br. J. Radiol.*, vol. 84, no. 1007, pp. 967–996, 2011.

[3] K. Fujimoto, et al., “Continuous-time method and its discretization to inverse problem of intensity-modulated radiation therapy treatment planning,” *Commun. Nonlinear Sci. Numer. Simul.*, vol. 19, no. 6, pp. 1996–2004, 2014.

[4] K. Kawamura, et al., “Fast computation method for total variation minimization,” *IEICE Trans. Inf. Syst.*, vol. 93, no. 3, pp. 326–335, 2010.