# Period-Doubling Bifurcation of Quasi-Periodic Solutions in Flow

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**Abstract—**
Depending on a Poincaré section, a period-doubling bifurcation of a 2-torus attractor in flow is observed as different two types bifurcations of ICCs on a Poincaré sections. We demonstrate this fact in the third-order Duffing equation with periodic external force system [1]. We clarify that this bifurcation by using Lyapunov exponents and Lyapunov bundles. The Lyapunov bundle is a set of generalized eigenvectors of a periodic solution to a each point of quasi-periodic solution.

## 1. Introduction

Bifurcation of 2-torus in flow is a very popular research theme in recent years. A 2-torus is one kind of quasi-periodic solutions in continuous dynamical systems (flow), in which two incommensurate frequencies exist. In general, to analyze bifurcation of flows, we take a Poincaré section, then the flow system can be converted to a map system. If a solution is a 2-torus in flow, it is observed as a closed curve (1-torus) on Poincaré section which is called Invariant Closed Curve (ICC). Depending on a Poincaré section of a flow, one can observe 2 types of curve doubling bifurcation of an ICC. At the type I, an ICC with one loop bifurcates to an ICC with two loops. At the type II, an ICC with one loop bifurcates to two ICCs. Due to Ashwin [2], these bifurcations are the same bifurcation in the original flow. Hence, we conjecture that there is only one doubling bifurcation in flow, and we named this bifurcation Double Covering (DC) bifurcatoin. In this study, we clarify DC bifurcation of a 2-torus in flow.

## 2. Local Bifurcations of ICCs

Fig. 1 represents the schematics of bifurcation types of ICCs; namely, Saddle-Node (SN), Pitchfork (PF), Transcritical (TC), Double Covering (DC), Period Doubling (PD), and Neimark-Sacker bifurcations of an ICCs. In SN bifurcation, a pair of stable and saddle ICCs coalesces and disappears. In PF bifurcation, a stable ICC is bifurcated to unstable ICC in addition to two stable ICCs. In TC bifurcation, a stable ICC and an unstable ICC exchange stability. In DC bifurcatoin, an ICC is bifurcated to a single

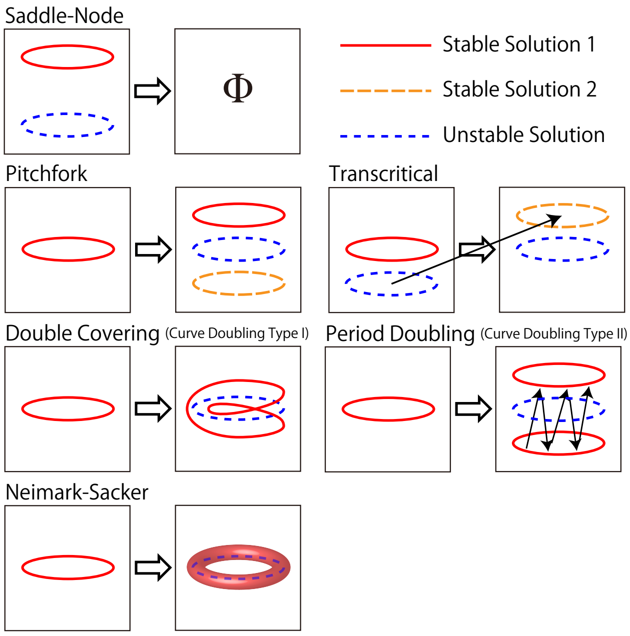



Figure 1: Local Bifurcations of an ICC.

twisted doubled ICC in addition to a saddle ICC[1], and in PD bifurcatoin, an ICC is bifurcated to one saddle ICC in addition to two stalbe ICCs that are permuted by map. In NS bifurcation, an ICC is bifurcated to one unstable ICC in addition to a stable 2-torus. We insist that the local bifurcation of an ICC are restricted to these bifurcations and no other bifurcation can occur. Moreover, due to Ashwin [2], if one observe DC or PD bifurcation on a Poincaré section of flow, these are the same in the original flow. Namely, depending on the position of Poincaré section, one can observe either DC or PD bifurcation for the same flow. In other words, there is only one curve doubling bifurcation in flow, and we call it DC. In Ashwin's paper, DC is called curve doubling type I, and PD is called curve doubling type II. In this study, we focus on these two types of curve doubling. Fig. 2 explains how to take 2 types Poincaré sections of 2-torus in flow. We name these 2 types of sections "Lon-

[1]The reason why we call this bifurcation DC, this bifurcation doubles the length of an ICC.

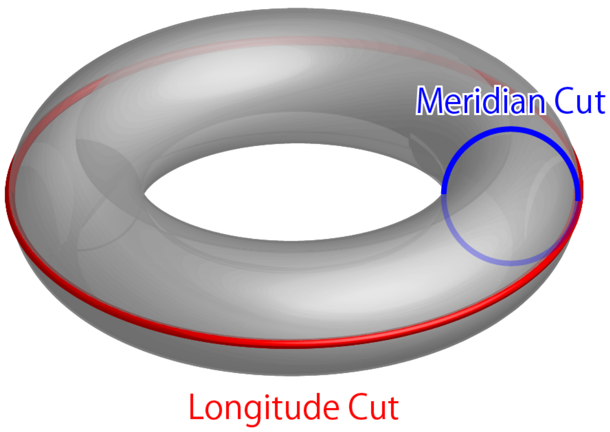



Figure 2: 2 types Poincaré sections of 2-torus in flow.

gitude Cut" and "Meridian Cut."

Recently, to research the bifurcation mechanism of ICC, we calculate the Lyapunov Vector (LV) [3] for continuous dynamical systems [4]. The LV is a generalization of eigenvector of a periodic solution to a quasi-periodic solution. The LVs are calculated at each points of an ICC. Therefore, on an ICC, the dense set of LVs become a bundle, and we name this bundle the Lyapunov Bundle (LB). In $n$-dimensional system, an ICC has $n$ LEs and corresponding $n$ LBs. We call non-zero Lyapunov Exponent (LE) which is closest to zero, Dominant LE (DLE) and we call coressponding LB Dominant LB (DLB). Moreover, we clarify that the local bifurcation of ICC can be clasified by the topology of DLB before the birfurcatoin point [5]. there are 3 types of topology of LBs: the Annulus ($A$), Mobius ($M$) and Focus ($F$) type. The $A$ type is orientable, and the others are non-orientable when not only the topology but also the tangent map are taken into consideration. Namely, the $A$ type is divided to 2 types: the $A^+$ and $A^-$ type. In the $A^+$ type LB, a LV is mapped to the same side of an ICC by tangent map. In the $A^-$ type LB, a LV is mapped the other side of an ICC by tangent map and toggle between each side of an ICC. In the $M$ type LB, the form of LB is Mobius band. In the $F$ type LB, a LV is mapped no an ICC with rotating and the shape of LB is like a Test-tube brush. The bifurcation of 2-torus in flow can be characterized with the combination of these two types of bifurcations, becase the ICC on a Poincaré section is 2-dimentional torus n the original flow. We have discovered that, when the DC bifuraction of 2-torus in flow is occured, the combinations of LBs associated with the ICC are the $A^- \times M$ or $M \times A^-$ or $M \times M$ type in map. We clarify that If one can observe the $A^-$ or $M$ type DLB on a Poincaré section of 2-torus in flow before the local bifurcation point, DC bifurcation of 2-torus in flow occurs at the local bifurcation point. In this study, to demonstrate the above-mentioned mechanism, we introduce an actual circuit's example by Kawakami et. al. [1].

## 3. Circuit System and Equation

The circuit equation we concern is the following third-order Duffing equation with periodic external force appeared in [1]. This equation is obtained from a resonant circuit containing saturable reactors with secondary d-c windings [6] p. 265, Eq. (11. 8).

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -k_1 y - 1/8(x^2 + 3z^2)x + B\cos t, \\ \dot{z} &= -1/8k_2(3x^2 + z^2)z + B_0. \end{cases} \quad (1)$$

For changing a Poincaré section, we will transform Eq. (1) to the following form

$$\begin{cases} \dot{x}_0 &= x_1, \\ \dot{x}_1 &= -k_1 x_1 - 1/8(x_0^2 + 3x_2^2)x_0 + Bx_3, \\ \dot{x}_2 &= -1/8k_2(3x_0^2 + x_2^2)x_2 + B_0, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= \mu(1 - x_3^2 - x_4^2)x_4 - x_3. \end{cases} \quad (2)$$

where $x_0 = x, x_1 = y$, and $x_2 = 2(z - 1.5)$. The solutions of the last two equations in Eq. (2) becomes "$x_3 = \cos t$" by giving the initial condition $x_3(0) = 1$ and $x_4(0) = 0$. Therefore, dynamical behavior of Eq. (2) is the same as Eq. (1). We fix the parameters in Eq. (2) as $k_1 = 0.03, k_2 = 0.05$ and $B_0 = 0.03$. In this system, a 2-torus attractor in flow exists at $B = 0.07$ as shown in Fig. 3 projected on the $x_0$-$x_1$-$x_2$ space.

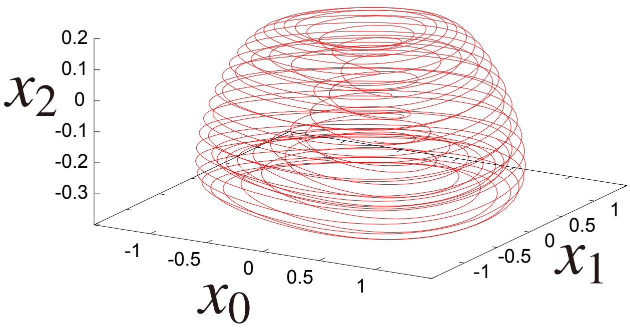



Figure 3: The 2-torus attractor in flow projected on the $x_0$-$x_1$-$x_2$ space. The parameters are $k_1 = 0.03, k_2 = 0.05, B_0 = 0.03$ and $B = 0.07$.

## 4. Double Covering Bifurcation and 2 types Poincaré Sectoins

Fig. 4 presents LEs of the 2-torus in terms of $B$. $M$ denotes the multiplicity of DLE. $M = 1$ is ommited for simplicity. As a results, At the point labeled by ①, NS bifurcation of 1-torus (periodic solution) occured. At the point labeled by ②, DC bifurcation of 2-torus (quasi-periodic solution) occured. We forcused on the DC bifurcation point labeled ②.

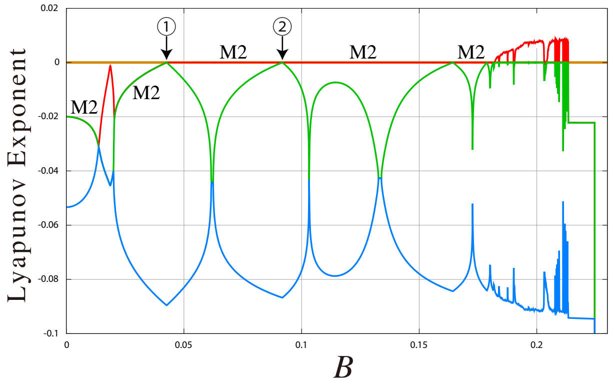
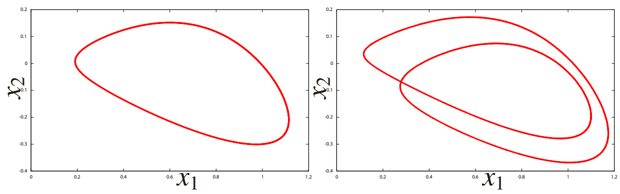



Figure 4: Lyapunov exponents of the attractor in Eq. (2) with the increase of $B$. Other parameters are the same as those in Fig. 3. As a result, the point labeled by ① presents Neimark-Sacker bifurcation of a 1-torus (periodic solution). and the point labeled by ② presents DC bifurcation of 2-torus (quasi-periodic slution).

On the Poincaré section which is the meridian cut, one can observe DC bifurcation of the ICC. Fig. 5(a) shows the attractor before the DC bifurcatoin which is an ICC with one loop. Fig. 5(b) shows the attractor after the DC bifurcatoin which is an ICC with two loops. Fig. 6 represents the DLB on the meridian cut. The DLB is twisted once on the ICC. Namely, this is cleary the $M$ type DLB.




(a) (b)

Figure 5: The ICC (projected to the $x_1$-$x_2$ space) observed on the Poincaré section which is the meridian cut. Parameters are the same as those in Fig. 3. (a) An ICC with one loop before the DC bifurcation for $B = 0.065$. (b) An ICC with two-loop after the DC bifurcation for $B = 0.070$.

On the another Poincaré section which is the longitude cut, one can observe PD bifurcation of the ICC at the same parameter. Fig. 7(a) shows the attractor before the DC bifurcatoin which is an ICC with one loop. Fig. 7(b) shows the attractor after the DC bifurcatoin which is two ICCs with one loop. Namely, there are two disjoint loops and the iterates successively toggle between them as shown in every other iteration map Fig. 7(c). Fig. 8 represents the DLB on the longitude cut. The DLB exists both side of the ICC and LV is mapped toggle between each side of ICC. Namely, this is cleary the $A^-$ type DLB.

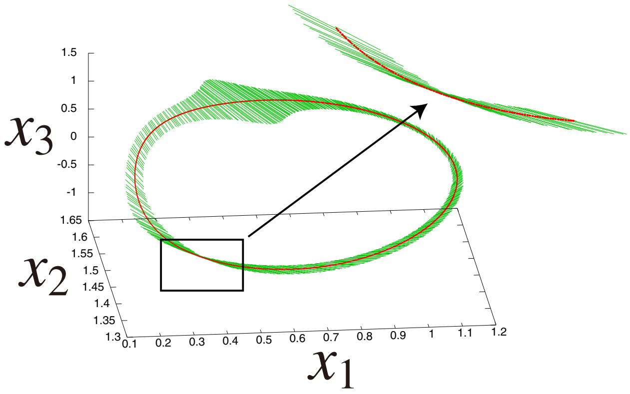



Figure 6: The the $M$ type DLB on the meridian cut. Red points present mapped points. A set of green lines present the DLB.

## 5. Conclusion

We demonstrate that one can observe different 2 type bifurcations of ICCs depending on Poincaré sections in the same 2-torus attractor in the original flow in the third-order Duffing equation with periodic external force system [1]. We clarify that the DC bifurcation of the 2-torus is the combination of 2 types LBs: the $A^- \times M$ type.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## References


[1] H. Kawakami and K. Kobayashi, "Branching of type i for almost periodic solutions in nonautonomous third order systems - an experimental result by analog simulation -(japanese)," *IEICE Tech. Rep.*, vol. 78, no. 174, pp. 27–36, Mar. 1978.

[2] P. Ashwin and J. W. Swift, "Torus doubling in four weakly coupled oscillators," *Int. J. Bifurcation and Chaos*, vol. 05, no. 01, pp. 231–241, Feb. 1995.

[3] F. Ginelli, P. Poggi, A. Turchi, H. Chaté, R. Livi, and A. Politi, "Characterizing dynamics with covariant lyapunov vectors," *Phys. Rev. Lett.*, vol. 99, p. 130601, Sep 2007.

[4] K. Kamiyama, M. Komuro, and T. Endo, "Algorithms for obtaining a saddle torus between two attractors,"

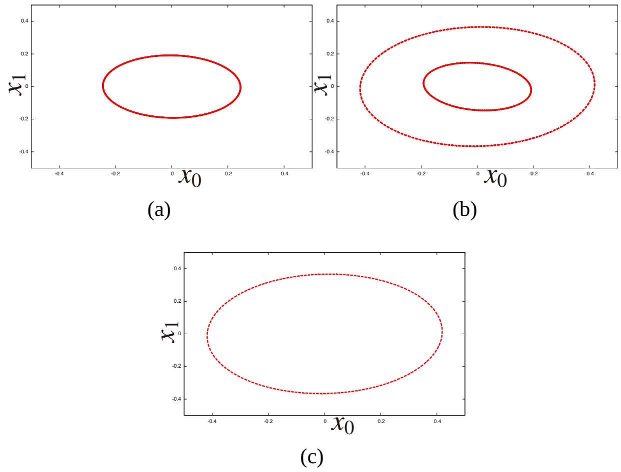


Figure 7: The ICC (projected to the $x_0$-$x_1$ space) observed on the Poincaré section which is the longitude cut. Parameters except $B$ are the same as those in Fig. 3. (a) An ICC with one loop before the PD bifurcation for $B = 0.065$. (b) A pair of disjoint ICCs with one loop after the PD bifurcation for $B = 0.07$. (c) An ICC with one loop for $B = 0.070$ for the every other iteration.


*Int. J. Bifurcation and Chaos*, vol. 23, no. 09, p. 1330032, Sep. 2013.

[5] ——, “Pattern categorization of the bifurcation of quasi-periodic solutions by using covariant lyapunov bundle (in japanese),” *IEICE Tech. Rep.*, vol. 113, no. 69, pp. 77–80, May 2013.

[6] C. Hayashi, *Nonlinear Oscillatinos in Physical Systems*. McGraw-Hill, 1964.


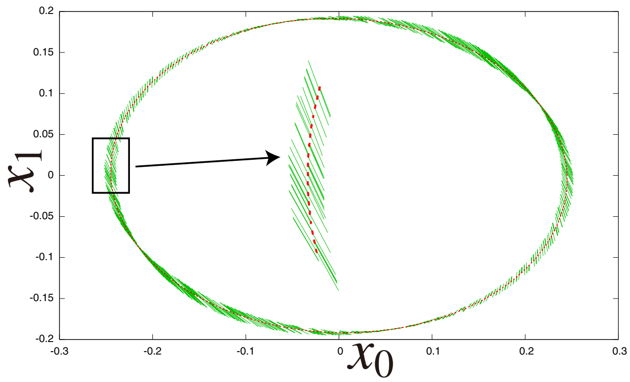


Figure 8: The the $A^{-}$ type DLB on the meridian cut. Red points present mapped points. A set of green lines present the DLB.